

Computing all maps into a sphere*

Martin Čadek^a Marek Krčál^{b,c} Jiří Matoušek^{b,c,d} Francis Sergeraert^e
Lukáš Vokřínek^a Uli Wagner^d

January 26, 2013

Abstract

We present an algorithm for computing $[X, Y]$, i.e., all homotopy classes of continuous maps $X \rightarrow Y$, where X, Y are topological spaces given as finite simplicial complexes, Y is $(d-1)$ -connected for some $d \geq 2$ (for example, Y can be the d -dimensional sphere S^d), and $\dim X \leq 2d - 2$. These conditions on X, Y guarantee that $[X, Y]$ has a natural structure of a finitely generated Abelian group, and the algorithm finds generators and relations for it. We combine several tools and ideas from homotopy theory (such as *Postnikov systems*, *simplicial sets*, and *obstruction theory*) with algorithmic tools from effective algebraic topology (*objects with effective homology*).

We hope that a further extension of the methods developed here will yield an algorithm for computing, in some cases of interest, the \mathbb{Z}_2 -index, which is a quantity playing a prominent role in Borsuk–Ulam style applications of topology in combinatorics and geometry, e.g., in topological lower bounds for the chromatic number of a graph. In a certain range of dimensions, deciding the embeddability of a simplicial complex into \mathbb{R}^d also amounts to a \mathbb{Z}_2 -index computation. This is the main motivation of our work.

We believe that investigating the computational complexity of questions in homotopy theory and similar areas presents a fascinating research area, and we hope that our work may help bridge the cultural gap between algebraic topology and theoretical computer science.

1 Introduction

The problem. One of the central themes in algebraic topology is understanding the structure of all *continuous* maps $X \rightarrow Y$, for given topological spaces X and Y (all maps between topological spaces in this paper are assumed to be continuous). For topological purposes, two maps $f, g: X \rightarrow Y$ are usually considered equivalent if they are *homotopic*, i.e.,

* The research by M. Č. and L. V. was supported by a Czech Ministry of Education grant (MSM 0021622409). The research by M. K. was supported by project GAUK 49209. The research by J. M. was partially supported by the ERC Advanced Grant No. 267165. The research by U. W. was supported by the Swiss National Science Foundation (SNF Project 200021-125309).

^aDepartment of Mathematics and Statistics, Masaryk University, Kotlářská 2, 611 37 Brno, Czech Republic

^bDepartment of Applied Mathematics, Charles University, Malostranské nám. 25, 118 00 Praha 1, Czech Republic

^cInstitute of Theoretical Computer Science (ITI), Charles University, Malostranské nám. 25, 118 00 Praha 1, Czech Republic

^dInstitute of Theoretical Computer Science, ETH Zurich, 8092 Zurich, Switzerland

^eInstitut Fourier, BP 74, 38402 St Martin, d'Hères Cedex, France

if one can be continuously deformed into the other¹; thus, the object of interest is $[X, Y]$, the set of all homotopy classes of maps $X \rightarrow Y$.

Many of the celebrated results throughout the history of topology can be cast as information about $[X, Y]$ for particular spaces X and Y . An early example is a famous theorem of Hopf from the 1930s, asserting that the homotopy class of a map $f: S^n \rightarrow S^n$, between two spheres of the same dimension, is in one-to-one correspondence with an integer parameter, the *degree* of f . Another great discovery of Hopf, with ramifications in modern physics and elsewhere, was a map $S^3 \rightarrow S^2$, now called by his name, that is not homotopic to a constant map.

These are early results in the theory of *higher homotopy groups*. For our purposes, the k th homotopy group $\pi_k(Y)$, $k \geq 2$, of a space Y can be thought of as the set $[S^k, Y]$ (which is, moreover, equipped with a suitable group operation).² In particular, the *homotopy groups of spheres* $\pi_k(S^n)$ are among the most puzzling objects of mathematics, and many respected papers have been devoted to computing them in special cases (see, e.g., the book [22]).

Related to the problem of determining $[X, Y]$ is the *extension problem*: given $A \subset X$ and a map $f: A \rightarrow Y$, can it be extended to a map $X \rightarrow Y$? For example, the famous *Brouwer fixed-point theorem* can be re-stated as non-extendability of the identity map $S^n \rightarrow S^n$ to the ball D^{n+1} . A number of topological concepts, which may look quite advanced and esoteric to a newcomer in algebraic topology, e.g. *Steenrod squares*, have a natural motivation in an attempt at a stepwise solution of the extension problem.

Earlier developments around the extension problems are described in Steenrod's paper [37] (based on a 1957 lecture series), which we can recommend, for readers with a moderate topological background, as an exceptionally clear and accessible, albeit somewhat outdated, introduction to this area. In that paper, Steenrod asks for an effective procedure for (some aspects of) the extension problem.

There has been an enormous amount of work in homotopy theory since the 1960s, with a wealth of new concepts and results, some of them opening completely new areas or reaching to distant branches of mathematics. However, as far as we could find out, the *algorithmic part* of the program discussed in [37] has not been explicitly completed up until now.

The only algorithmic paper concerning the computation of $[X, Y]$ we are aware of is that by Brown [2] from 1957(!). Brown showed that $[X, Y]$ is computable under the assumption that Y is 1-connected³ and all the higher homotopy groups $\pi_k(Y)$, $2 \leq k \leq \dim X$, are *finite* (this is a rather strong assumption, *not* satisfied by spheres, for example). Then he went on to show the *computability of the higher homotopy groups* $\pi_k(Y)$, $k \geq 2$, for every 1-connected Y . To do this, he overcame the problem of infinite homotopy groups (which we will discuss below) by a somewhat ad-hoc method, which does not seem to generalize to the $[X, Y]$ setting.

On the negative side, it is well known that the problem of computing $[X, Y]$, in full generality, is *algorithmically unsolvable*. Indeed, for Y connected, $[S^1, Y]$ is nontrivial exactly if $\pi_1(Y) \neq 0$, where $\pi_1(Y)$ is the *fundamental group* of Y , and the undecidability of $\pi_1(Y) \neq 0$ is a celebrated result of Adjan and of Rabin (see, e.g., the survey by Soare [34]). Actually,

¹More precisely, f and g are defined to be homotopic, in symbols $f \sim g$, if there is a continuous $F: X \times [0, 1] \rightarrow Y$ such that $F(\cdot, 0) = f$ and $F(\cdot, 1) = g$. With this notation, $[X, Y] = \{[f] : f: X \rightarrow Y\}$, where $[f] = \{g : g \sim f\}$ is the *homotopy class* of f .

²Strictly speaking, the isomorphism $\pi_k(Y) \cong [S^k, Y]$ needs mild assumptions on Y ; e.g., it holds if Y is a path-connected CW-complex.

³A k -connected space Y is one whose first k homotopy groups vanish; in other words, every map $S^i \rightarrow Y$ can be extended to D^{i+1} , the ball bounded by the S^i , $0 \leq i \leq k$.

this is the only hardness result known to us.⁴ For undecidability results concerning numerous more loosely related problems we refer to [34], [21], [20] and references therein.

Effective algebraic topology. In the 1990s, three independent collections of works appeared with the goal of making various more advanced methods of algebraic topology *effective* (algorithmic): by Schön [28], by Smith [33], and by Rubio, Sergeraert, Dousson, and Romero (e.g., [31, 24, 23, 25]; also see [27] for an exposition). These obtain general *computability* results, and in the case of Rubio et al., a *practical implementation* as well, but none of them provides any running time bounds.

Roughly speaking, Rubio et al. provide algorithms that can construct basic topological spaces, such as finite simplicial complexes or *Eilenberg–MacLane spaces* (discussed below), and then obtain new spaces from them by various operations, e.g., the Cartesian product, the *loop space* and the *bar construction*, the *total space of a fibration*, etc. Interestingly, and crucially, these objects and constructions are often of infinitary nature, which means that the resulting spaces have to be represented in a certain implicit manner. Yet one can compute homology and cohomology groups (of given dimensions) of the resulting objects; one speaks of *objects with effective homology*.

The problem of computing $[X, Y]$ and the extension problem were not addressed in those papers, but we build on them to some extent, relying on objects with effective homology for implementing certain operations in our algorithm.

Our work. We are generally interested in the *computational complexity* of the problem of computing $[X, Y]$. We assume that X and Y are given as finite simplicial complexes (or, more generally, *simplicial sets* with finitely many nondegenerate simplices, as discussed below).

We would like to find, on the one hand, sufficient conditions on X and Y , as weak as possible, making the problem decidable, or even polynomial-time solvable, and on the other hand, interesting settings where the problem can be proved algorithmically intractable (undecidable or NP-hard, say). We also believe that similar methods may bring results for the extension problem and for other related questions.

Here we prove the following positive result:

Theorem 1.1. *Let $d \geq 2$. Assuming that $Y = S^d$ or, more generally, that Y is $(d - 1)$ -connected, and that $\dim X \leq 2d - 2$, the set $[X, Y]$ is computable, in the following sense: It is known that, under the above conditions on X and Y , $[X, Y]$ can be naturally endowed with a structure of a (finitely generated) Abelian group, in an essentially unique way.⁵ The algorithm computes the structure of this group (i.e., expresses it as a direct product of cyclic groups). Moreover, given a simplicial map $f: X \rightarrow Y$, it can be identified as an element of $[X, Y]$ (expressed in terms of generators), and consequently, it is possible to test homotopy of simplicial maps $X \rightarrow Y$.*

We establish Theorem 1.1 mainly by combining ideas and tools that have been essentially known. We see our main contribution as that of synthesis: identifying suitable methods,

⁴There is also a result of Anick [1] on #P-hardness of computing the higher homotopy groups. However, the way he presents it, it is not immediately relevant for spaces given as simplicial complexes, since his reduction uses a very compact representation of the input space—roughly speaking, he needs to encode degrees of attaching maps as binary integers. Perhaps with some more work one could also use his method to show hardness of computing $\pi_k(Y)$ for Y given as a simplicial complex, say.

⁵In particular, the groups $[X, S^d]$ are known as the *cohomotopy groups* of X ; see [14]. More precisely, cohomotopy groups are defined using *pointed* maps, i.e., maps sending a distinguished point of X to a distinguished point of S^d . But under the conditions of the theorem, the pointed and non-pointed cases are equivalent.

putting them all together, and organizing the result in a hopefully accessible way, so that it can be built on in the future. Some technical steps are apparently new; in this direction, our main technical contribution is probably a suitable implementation of the group operation on $[X, Y]$ and recursive testing of nullhomotopy.

Applications, motivation. We consider the fundamental nature of the algorithmic problem of computing $[X, Y]$ a sufficient motivation of our research (e.g., because $[X, Y]$ is indeed one of the most basic objects of study in algebraic topology). However, we also believe that work in this area will bring various connections and applications, also in other fields, possibly including practically usable software, e.g., for aiding research in topology.

A nice concrete application comes from the paper by Franek et al. [7]. They provide an algorithm testing if a given system of equations involving analytic functions has a “robust zero”, and in order to extend their result to more general situations, they ask for an algorithm testing *nullhomotopy* (i.e., homotopy to a constant map) of a map into S^d . Our Theorem 1.1 provides such an algorithm in a certain range of dimensions.

Our motivation for starting this project was the computation of the \mathbb{Z}_2 -index (or *genus*) $\text{ind}(X)$ of a \mathbb{Z}_2 -space⁶ X , i.e., the smallest d such that X can be equivariantly mapped into S^d . We hope that by extending the methods of the present paper, one can obtain an algorithm for deciding whether $\text{ind}(X) \leq d$, provided that $\dim(X) \leq 2d - 2$.

The problem of computing $\text{ind}(X)$ arises, among others, in the problem of *embeddability* of topological spaces, which is a classical and much studied area (see, e.g., the survey by Skopenkov [32]). One of the basic questions here is, given a k -dimensional finite simplicial complex K , can it be (topologically) embedded in \mathbb{R}^d ? The celebrated *Haefliger–Weber theorem* from the 1960s asserts that, in the *metastable range of dimensions*, i.e., for $k \leq \frac{2}{3}d - 1$, embeddability is equivalent to $\text{ind}(K_\Delta^2) \leq d - 1$, where K_Δ^2 is a certain \mathbb{Z}_2 -space constructed from K (the *deleted product*). Thus, in this range, the embedding problem is, computationally, a special case of \mathbb{Z}_2 -index computation; see [17] for a study of algorithmic aspects of the embedding problem, where the metastable range was left as one of the main open problems.

The \mathbb{Z}_2 -index also appears as a fundamental quantity in combinatorial applications of topology. For example, the celebrated result of Lovász on Kneser’s conjecture can nowadays be re-stated as $\chi(G) \geq \text{ind}(B(G)) + 2$, where $\chi(G)$ is the chromatic number of a graph G , and $B(G)$ is a certain simplicial complex constructed from G (see, e.g., [16]). We find it striking that *nothing* seems to be known about the computability of such an interesting quantity as $\text{ind}(B(G))$. Indeed, some authors (e.g., Kozlov [15]) consider a weaker, cohomologically defined index, in part because of the suspected intractability of the \mathbb{Z}_2 -index.

Further work. Besides the problem of adapting the machinery behind Theorem 1.1 to the equivariant setting, or to the setting of the extension problem, there are number of other open questions related to our work.

Polynomiality. At present we do not state any bounds on the running time of the algorithm. A critical part for the running time are subroutines for building a *Postnikov system* of Y and evaluating *Postnikov classes*. For this, one can use algorithms sketched in [26, 25], but these appear to be at least exponential. More precisely, we believe that all of the steps in these algorithms are polynomial for *fixed* dimension, with the single exception of an *effective*

⁶A \mathbb{Z}_2 -space is a topological space X with an action of the group \mathbb{Z}_2 ; the action is described by a homeomorphism $\nu: X \rightarrow X$ with $\nu \circ \nu = \text{id}_X$. A primary example is a sphere S^d with the antipodal action $x \mapsto -x$. An *equivariant map* between \mathbb{Z}_2 -spaces is a continuous map that commutes with the \mathbb{Z}_2 actions.

homology reduction for the simplicial Eilenberg–MacLane space $K(\mathbb{Z}, 1)$ (see, e.g., [27] for these notions; some of them are also discussed later in the present paper).

Some of us have obtained preliminary results indicating that effective homology of $K(\mathbb{Z}, 1)$ should be computable in polynomial time as well. We plan to present these results in a follow-up paper, together with a self-contained description of the Postnikov system algorithm. The hoped-for outcome should be, in the setting of Theorem 1.1, a running time polynomial in the size of X and Y for every *fixed* d . On the other hand, we consider polynomial dependence on d highly unlikely—for example, because Theorem 1.1 includes the computation of the *stable* homotopy groups $\pi_{d+k}(S^d)$, $k \leq d - 2$; these are unlikely to be easily computable, in view of their notorious mathematical difficulty.

Hardness? We suspect that once the assumptions in Theorem 1.1 are weakened, the problem of deciding, say, nontriviality of $[X, Y]$ may become intractable. This is, in our opinion, one of the most interesting open problems related to our work.

Explicit maps. The algorithm works with certain implicit representations of the elements of $[X, Y]$, it can output a set of generators of the group in this representation, and it contains a subroutine implementing the group operation. However, converting these implicit representations into actual maps $X \rightarrow Y$ (given, say, as simplicial maps from a sufficiently fine subdivision of X into Y) looks problematic, and even if worked out, it seems unlikely to yield any reasonable bounds on the complexity of the resulting explicit maps.

General remarks. *Algorithmic* or *computational topology* has been a blooming discipline in recent years (see, e.g., [4, 40]). Our work addresses issues different from those investigated in the current mainstream of this field. Computations with simplicial complexes of arbitrary dimension (as opposed to specific questions in dimension 2 or 3) usually concern *homology* (for example, the `polymake` software [9] contains a module for computing homology), which has been regarded as an algorithmic tool since its origins. We study *homotopic* questions, generally regarded as much less tractable.

Although such questions have been thoroughly studied from a topological perspective already in the 1950s and 1960s, we are not aware of any work in this direction in theoretical computer science, with the perspective of computational complexity. We believe that questions similar to those studied here offer an exciting field for complexity-theoretic study.

On the one hand, there is an enormous topological literature with many beautiful ideas; indeed, in our experience, a problem with algorithmization may sometimes be an *abundance* of topological results, and the need to sort them out. On the other hand, the classical computational tools have been mostly designed for the “paper-and-pencil” model of calculation, where a calculating mathematician can, e.g., easily switch between different representations of an object or fill in some missing information by clever ad-hoc reasoning. Adapting the various methods to machine calculation may need a different approach; for instance, a recursive formulation may be preferable to an explicit, but cumbersome, formula (see, for example, [24, 30] for an explanation of algorithmic difficulties with *spectral sequences*, a basic and powerful computational tool in topology).

We aim at accessibility of our presentation to a general computer science audience with only a moderate topological background, in order to help bridge the current “cultural gap” between computer science and topology.

An outline of the methods. In the rest of this section, we sketch the main ideas and tools in the algorithm. Some topological notions are left undefined here; we will introduce

them later.

Conceptually, the basis of the algorithm is classical *obstruction theory* [5]. For a first encounter, it is probably easier to consider a version of obstruction theory which proceeds by constructing maps $X \rightarrow Y$ inductively on the i -dimensional *skeleta*⁷ of X , extending them one dimension at a time. (For the actual development, we use a different version of obstruction theory, where we lift maps from X through stages of a Postnikov system of Y .)

In a nutshell, at each stage, the extendability of a map from the i -skeleton to the $(i + 1)$ -skeleton is characterized by vanishing of a certain *obstruction*, which can, more or less by known techniques, be evaluated algorithmically.

Textbook expositions may give the impression that obstruction theory is a general algorithmic tool for testing the extendability of maps (this is actually what some of the topologists we consulted seemed to assume). However, the extension at each step is generally not unique, and extendability at higher stages may depend, in a nontrivial way, on the choices made earlier. Thus, in principle, one needs to search an infinitely branching tree of extensions.⁸

In our setting, we make essential use of the group structure on the sets $[X, Y]$ (mentioned in Theorem 1.1), as well as on some related ones, for a finite encoding of the set of all possible extensions at a given stage.

The description of our algorithm has several levels. On the top level, we talk about operations on Abelian groups, whose elements are homotopy classes of maps (and we need to be careful in distinguishing “how explicitly” the relevant groups are available to us). On a lower level, the group operation and other primitives are implemented by computations with *concrete representatives* of the homotopy classes; interestingly, on the level of the representatives, the operations are generally non-associative.

The space Y enters the computation in the form of a *Postnikov system*. This is a topological concept from the 1950s (usually considered unsuitable for concrete computations by topologists; see, e.g., [13]); roughly speaking, it provides a way of building Y from “canonical pieces”, called *Eilenberg–MacLane spaces*, whose homotopy structure is the simplest possible, although they are not that simple combinatorially.

Our main data objects are *simplicial sets*, an ingenious generalization of simplicial complexes. They are suitable for algorithmic representation of Eilenberg–Mac Lane spaces and other infinite objects in the algorithm. The stages P_i of the Postnikov system are built as simplicial sets in such a way that every *continuous* map $X \rightarrow P_i$ is homotopic to a *simplicial* map. The proof of Theorem 1.1 will rely on two facts: that for $\dim X \leq 2d - 2$, there is an isomorphism $[X, Y] \cong [X, P_{2d-2}]$, and that we can compute $[X, P_i]$ inductively for $i \leq 2d - 2$.

Then, due to the properties of the Eilenberg–MacLane spaces, simplicial maps into P_i can be compactly represented by certain sequences of *cochains* on X . Concretely, a map appears in the algorithm as a labeling of the simplices of X by elements of various Abelian groups.

An important component of the algorithm are subroutines, not treated in detail in this paper, for evaluating k_i ’s, the i th *Postnikov classes* of Y , $d \leq i \leq 2d - 2$. The input to k_i is represented as a simplex with faces labeled by elements of appropriate Abelian groups, and the output lies in yet another Abelian group.

For Y fixed, these subroutines can be hard-wired once and for all. In some particular cases, they are given by known explicit formulas. In particular, for $Y = S^d$, k_d corresponds to the

⁷The k -skeleton of a simplicial complex X consists of all simplices of X of dimension at most k .

⁸Brown’s result mentioned earlier, on computing $[X, Y]$ with the $\pi_k(Y)$ ’s finite, is based on a complete search of this tree, where the assumptions on Y guarantee the branching to be finite.

famous *Steenrod square* [36, 37] (more precisely, to the reduction from integral cohomology to mod 2 cohomology followed by the Steenrod square Sq^2), and k_{d+1} to *Adem's secondary cohomology operation*.⁹ However, in the general case, the only way of evaluating the k_i we are aware of is using *objects with effective homology* mentioned earlier. In this context, our result can also be regarded as an algorithmization of certain *higher cohomology operations* (see, e.g., [19]), although our development of the required topological underpinning is somewhat different and, in a way, simpler.¹⁰

2 Operations with Abelian groups

On the top level, our algorithm works with finitely generated Abelian groups. The structure of such groups is simple (they are direct sums of cyclic groups) and well known, but we will need to deal with certain subtleties in their algorithmic representations.

In our setting, an Abelian group A is represented by a set \mathcal{A} , whose elements are called *representatives*; we also assume that the representatives can be stored in a computer. For $\alpha \in \mathcal{A}$, let $[\alpha]$ denote the element of A represented by α . The representation is generally non-unique; we may have $[\alpha] = [\beta]$ for $\alpha \neq \beta$.

We call A represented in this way *semi-effective* if algorithms for the following three tasks are available:

- (SE1) Provide an element $o \in \mathcal{A}$ representing the neutral element $0 \in A$.
- (SE2) Given $\alpha, \beta \in \mathcal{A}$, compute an element $\alpha \boxplus \beta \in \mathcal{A}$ with $[\alpha \boxplus \beta] = [\alpha] + [\beta]$ (where $+$ is the group operation in A).
- (SE3) Given $\alpha \in \mathcal{A}$, compute an element $\boxminus \alpha \in \mathcal{A}$ with $[\boxminus \alpha] = -[\alpha]$.

We stress that as a binary operation on \mathcal{A} , \boxplus is not necessarily a group operation; e.g., we may have $\alpha \boxplus (\beta \boxplus \gamma) \neq (\alpha \boxplus \beta) \boxplus \gamma$, although of course, $[\alpha \boxplus (\beta \boxplus \gamma)] = [(\alpha \boxplus \beta) \boxplus \gamma]$.

For a semi-effective Abelian group, we are generally unable to decide, for $\alpha, \beta \in \mathcal{A}$, whether $[\alpha] = [\beta]$ (and, in particular, to certify that some element is nonzero).

Even if such an *equality test* is available, we still cannot infer much global information about the structure of A . For example, without additional information we cannot certify that A is infinite cyclic—it could always be large but finite cyclic, no matter how many operations and tests we perform.

We now introduce a much stronger notion, with all the structural information explicitly available. We call a semi-effective Abelian group A *fully effective* if it is finitely generated

⁹It is worth remarking that the k_i 's represent a “nonlinear part” of the algorithm, which otherwise, on the bottom level, deals mostly with solving systems of linear Diophantine equations. For example, a Steenrod square can be thought of as a quadratic form.

¹⁰Let us also mention the paper by Gonzales et al. [11], which provides algorithms for calculating certain primary and secondary cohomology operations on a finite simplicial complex (including the Steenrod square Sq^2 and Adem's secondary cohomology operation). But both their goal and the approach are different from ours. The algorithms in [11] are based on explicit combinatorial formulas for these operations on the cochain level. The goal is to speed up the “obvious” way of computing the image of a given cohomology class under the considered operation. In our setting, we have no general explicit formulas available, and also we can only work with the cohomology classes “locally” (since they are usually defined on infinite simplicial sets). That is, a cohomology class is represented by a cocycle, and that cocycle is given as an algorithm that can compute the value of the cocycle on any given simplex.

and we have an explicit expression of A as a direct sum of cyclic groups. More precisely, we assume that the following are explicitly available:

- (FE1) A list of generators a_1, \dots, a_k of A (given by representatives $\alpha_1, \dots, \alpha_k \in \mathcal{A}$) and a list (q_1, \dots, q_k) , $q_i \in \{2, 3, 4, \dots\} \cup \{\infty\}$, such that each a_i generates a cyclic subgroup of A of order q_i , $i = 1, 2, \dots, k$, and A is the direct sum of these subgroups.
- (FE2) An algorithm that, given $\alpha \in \mathcal{A}$, computes a representation of $[\alpha]$ in terms of the generators; that is, it returns $(z_1, \dots, z_k) \in \mathbb{Z}^k$ such that $[\alpha] = \sum_{i=1}^k z_i a_i$.

First we observe that, for full effectivity, it is enough to have A given by arbitrary generators and relations. That is, we consider a semi-effective A together with a list b_1, \dots, b_n of generators of A (again explicitly given by representatives) and an $m \times n$ integer matrix U specifying a complete set of relations for the b_i ; i.e., $\sum_{i=1}^n z_i b_i = 0$ holds iff (z_1, \dots, z_n) is an integer linear combination of the rows of U . Moreover, we have an algorithm as in (FE2) that allows us to express a given element a as a linear combination of b_1, \dots, b_n (here the expression may not be unique).

Lemma 2.1. *A semi-effective A with a list of generators and relations as above can be converted to a fully effective Abelian group.*

Proof. This amounts to a computation of a Smith normal form, a standard step in computing integral homology groups, for example (see [38] for an efficient algorithm and references).

Concretely, the Smith normal form algorithm applied on U yields an expression $D = SUT$ with D diagonal and S, T square and invertible (everything over \mathbb{Z}). Letting $\mathbf{b} = (b_1, \dots, b_n)$ be the (column) vector of the given generators, we define another vector $\mathbf{a} = (a_1, \dots, a_n)$ of generators by $\mathbf{a} := T^{-1}\mathbf{b}$. Then $D\mathbf{a} = 0$ gives a complete set of relations for the a_i (since $DT^{-1} = SU$ and the row spaces of SU and of U are the same). Omitting the generators a_i such that $|d_{ii}| = 1$ yields a list of generators as in (FE1). \square

In the remainder of this section, the special form of the generators as in (FE1) will bring no advantage—on the contrary, it would make the notation more cumbersome. We thus assume that, for the considered fully effective Abelian groups, we have a list of generators and an arbitrary integer matrix specifying a complete set of relations among the generators.

Locally effective mappings. Let X, Y be sets. We call a mapping $\varphi: X \rightarrow Y$ *locally effective* if there is an algorithm that, given an arbitrary $x \in X$, computes $\varphi(x)$.

Next, for semi-effective Abelian groups A, B , with sets \mathcal{A}, \mathcal{B} of representatives, respectively, we call a mapping $f: A \rightarrow B$ *locally effective* if there is a locally effective mapping $\varphi: \mathcal{A} \rightarrow \mathcal{B}$ such that $[\varphi(\alpha)] = f([\alpha])$ for all $\alpha \in \mathcal{A}$. In particular, we speak of a *locally effective homomorphism* if f is a group homomorphism.

Lemma 2.2 (Kernel). *Let $f: A \rightarrow B$ be a locally effective homomorphism of fully effective Abelian groups. Then $\ker(f) = \{a \in A : f(a) = 0\}$ can be represented as fully effective.*

Proof. This essentially amounts to solving a homogeneous system of linear equations over the integers.

Let a_1, \dots, a_m be a list of generators of A and U a matrix specifying a complete set of relations among them, and similarly for B , b_1, \dots, b_n , and V . For every $i = 1, 2, \dots, m$, we express $f(a_i) = \sum_{j=1}^n z_{ij} b_j$; then the $m \times n$ matrix $Z = (z_{ji})$ represents f in the sense that,

for $a = \sum_{i=1}^m x_i a_i$, we have $f(a) = \sum_{j=1}^n y_j b_j$ with $\mathbf{y} = \mathbf{x}Z$, where $\mathbf{x} = (x_1, \dots, x_m)$ and $\mathbf{y} = (y_1, \dots, y_n)$ are regarded as *row* vectors.

Since V is the matrix of relations in B , $\sum_{j=1}^n y_j b_j$ equals 0 in B iff $\mathbf{y} = \mathbf{w}V$ for an integer (row) vector \mathbf{w} . So $\ker f = \{\sum_i x_i a_i : \mathbf{x} \in \mathbb{Z}^m, \mathbf{x}Z = \mathbf{w}V \text{ for some } \mathbf{w} \in \mathbb{Z}^n\}$.

Given a system of homogeneous linear equations over \mathbb{Z} , we can use the Smith normal form to find a system of generators for the set of all solutions (see, e.g., [29, Chapter 5]). In our case, dealing with the system $\mathbf{x}Z = \mathbf{w}V$, we can thus compute integer vectors $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(\ell)}$ such that the elements $a'_k := \sum_{i=1}^m x_i^{(k)} a_i$, $k = 1, 2, \dots, \ell$, generate $\ker f$. By similar (and routine) considerations, which we omit, we can then compute a complete set of relations for the generators a'_k , and finally we apply Lemma 2.1. \square

The next operation is the dual of taking a kernel, namely, factoring a given Abelian group by the image of a locally effective homomorphism. For technical reasons, when applying this lemma later on, we will need the resulting factor group to be equipped with an additional algorithm that returns a “witness” for an element being zero.

Lemma 2.3 (Cokernel). *Let A, B be fully effective Abelian groups with sets of representatives \mathcal{A}, \mathcal{B} , respectively, and let $f: A \rightarrow B$ be a locally effective homomorphism. Then we can obtain a fully effective representation of the factor group $C := \text{coker}(f) = B/\text{im}(f)$, again with the set \mathcal{B} of representatives. Moreover, there is an algorithm that, given a representative $\beta \in \mathcal{B}$, tests whether β represents 0 in C , and if yes, returns a representative $\alpha \in \mathcal{A}$ such that $[f(\alpha)] = [\beta]$ in B .*

We remark that, as will become apparent from the proof, the assumption that A is fully effective is not really necessary. Indeed, all that is needed is that A be semi-effective and that we have an explicit list of (representatives of) generators for A . In order to avoid burdening the reader with yet another piece of terminology, however, we refrain from defining a special name for such representations.

Proof of Lemma 2.3. As a semi-effective representation for C , we simply reuse the one we already have for B . That is, we reuse \mathcal{B} (and the same algorithms for (SE1–3)) to represent the elements of C as well. To distinguish clearly between elements in B and in C , for $\beta \in \mathcal{B}$, we use the notation $b = [\beta]$ in B and $\bar{b} = [\bar{\beta}]$ for the corresponding element $b + \text{im}(f)$ in C .

For a fully effective representation of C , we need the following, by Lemma 2.1: first, a complete set of generators for C (given by representatives); second, an algorithm as in (FE2) that expresses an arbitrary element of C (given as $\beta \in \mathcal{B}$) as a linear combination of the generators; and, third, a complete set of relations among the generators.

For the first two tasks, we again reuse the solutions provided by the representation for B . Suppose b_1, \dots, b_n (represented by β_1, \dots, β_n) generate B . Then $\bar{b}_1, \dots, \bar{b}_n$ (with the same representatives) generate C . Moreover, by assumption, we have an algorithm that, given $\beta \in \mathcal{B}$, computes integers z_i such that $[\beta] = z_1 b_1 + \dots + z_n b_n$ in B ; then $[\bar{\beta}] = z_1 \bar{b}_1 + \dots + z_n \bar{b}_n$ in C .

A complete set of relations among the generators of C is obtained as follows. Let the matrix V specify a complete set of relations among the generators b_j of B , let a_1, \dots, a_m be a complete list of generators for A , and let Z be an integer matrix representing the homomorphism f with respect to the generators a_1, \dots, a_m and b_1, \dots, b_n as in the proof of Lemma 2.2. Then

$$U := \begin{pmatrix} Z \\ V \end{pmatrix}$$

specifies a complete set of relations among the \bar{b}_j in C . To see that this is the case, consider an integer (row) vector $\mathbf{y} = (y_1, \dots, y_n)$ and $\bar{b} := \sum_{j=1}^n y_j \bar{b}_j$. Then $\bar{b} = 0$ in C iff $b := \sum_{j=1}^n y_j b_j \in \text{im}(f)$, i.e., iff there exists an element $a = \sum_{i=1}^m x_i a_i \in A$ such that $b - f(a) = 0$ in B . By definition of Z and by assumption on V , this is the case iff there are integer vectors \mathbf{x} and \mathbf{x}' such that $\mathbf{y} = \mathbf{x}Z + \mathbf{x}'V$, an integer combination of rows of U .

It remains to prove the second part of Lemma 2.3, i.e., to provide an algorithm that, given $\beta \in \mathcal{B}$, tests whether $[\bar{\beta}] = 0$ in C , or equivalently, whether $[\beta] \in \text{im}(f)$, and if so, computes a preimage. For this, we express $[\beta] = \sum_{j=1}^n y_j \bar{b}_j$ as an integer linear combination of generators of B and then solve the system $\mathbf{y} = \mathbf{x}Z + \mathbf{x}'V$ of integer linear equations as above (where we rely again on Smith normal form computations). \square

The last operation is conveniently described using a *short exact sequence* of Abelian groups:

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \quad (1)$$

(in other words, we assume that $f: A \rightarrow B$ is an injective homomorphism, $g: B \rightarrow C$ is a surjective homomorphism, and $\text{im } f = \ker g$). It is well known that the middle group B is determined, up to isomorphism, by A, C, f , and g . For computational purposes, though, we also need to assume that the injectivity of f is “effective”, i.e., witnessed by a locally effective inverse mapping r , and similarly for the surjectivity of g . This is formalized in the next lemma.

Lemma 2.4 (Short exact sequence). *Let (1) be a short exact sequence of Abelian groups, where A and C are fully effective, B is semi-effective, $f: A \rightarrow B$ and $g: B \rightarrow C$ are locally effective homomorphisms, and suppose that, moreover, the following locally effective maps (typically not homomorphisms) are given:*

- (i) $r: \text{im } f = \ker g \rightarrow A$ such that $f(r(b)) = b$ for every $b \in B$ with $g(b) = 0$.¹¹
- (ii) A map of representatives¹² $\xi: \mathcal{C} \rightarrow \mathcal{B}$ (where \mathcal{B}, \mathcal{C} are the sets of representatives for B, C , respectively) that behaves as a section for g , i.e., such that $g([\xi(\gamma)]) = [\gamma]$ for all $\gamma \in \mathcal{C}$.

Then we can obtain a fully effective representation of B .

Proof. Let a_1, \dots, a_m be generators of A and c_1, \dots, c_n be generators of C , with fixed representative $\gamma_j \in \mathcal{C}$ for each c_j . We define $b_j := [\xi(\gamma_j)]$ for $1 \leq j \leq n$.

Given an arbitrary element $b \in B$, we set $c := g(b)$, express $c = \sum_{j=1}^n z_j c_j$, and let $b^* := b - \sum_{j=1}^n z_j b_j$. Since $g(b^*) = g(b) - \sum_{j=1}^n z_j g(b_j) = 0$, we have $b^* \in \ker g$, and so $a := r(b^*)$ is well defined. Then we can express $a = \sum_{i=1}^m y_i a_i$, and we finally get $b = \sum_{i=1}^m y_i f(a_i) + \sum_{j=1}^n z_j b_j$.

Therefore, $(f(a_1), \dots, f(a_m), b_1, \dots, b_n)$ is a list of generators of B , computable in terms of representatives, and the above way of expressing b in terms of generators is algorithmic. Moreover, we have $b = 0$ iff $g(b) = 0$ and $r(b) = 0$, which yields equality test in B .

¹¹The equality $f(r(b)) = b$ is required on the level of group elements, and not necessarily on the level of representatives; that is, it may happen that $\varphi(\rho(\beta)) \neq \beta$, although necessarily $[\varphi(\rho(\beta))] = [\beta]$, where φ represents f and ρ represents r .

¹²For technical reasons, in the setting where we apply this lemma later, we do not get a well-defined map $s: C \rightarrow B$ on the level of group elements, that is, we cannot guarantee that $[\gamma_1] = [\gamma_2]$ implies $[\xi(\gamma_1)] = [\xi(\gamma_2)]$. Because of the injectivity of f , this problem does not occur for the map r .

It remains to determine a complete set of relations for the described generators (and then apply Lemma 2.1). Let U be a matrix specifying a complete set of relations among the generators a_1, \dots, a_m in A , and V is an appropriate matrix for c_1, \dots, c_n .

Let (v_{k1}, \dots, v_{kn}) be the k th row of V . Since $\sum_{j=1}^n v_{kj}c_j = 0$, we have $b_k^* := \sum_{j=1}^n v_{kj}b_j \in \ker g$, and so, as above, we can express $b_k^* = \sum_{i=1}^m y_{ik}f(a_i)$. Thus, we have the relation $-\sum_{i=1}^m y_{ik}f(a_i) + \sum_{j=1}^n v_{kj}b_j = 0$ for our generators of B .

Let $Y = (y_{ik})$ be the matrix of the coefficients y_{ik} constructed above. We claim that the matrix

$$\begin{pmatrix} -Y & V \\ U & 0 \end{pmatrix}$$

specifies a complete set of relations among the generators $f(a_1), \dots, f(a_m), b_1, \dots, b_n$ of B . Indeed, we have just seen that the rows in the upper part of this matrix correspond to valid relations, and the relations given by the rows in the bottom part are valid because U specifies relations among the a_i in A and f is a homomorphism.

Finally, let

$$x_1f(a_1) + \dots + x_mf(a_m) + z_1b_1 + \dots + z_nb_n = 0 \quad (2)$$

be an arbitrary valid relation among the generators. Applying g and using $g \circ f = 0$, we get that $\sum_{j=1}^n z_jc_j = 0$ is a relation in C , and so (z_1, \dots, z_n) is a linear combination of the rows of V .

Let (w_1, \dots, w_m) be the corresponding linear combination of the rows of $-Y$. Then we have $\sum_{i=1}^m w_if(a_i) + \sum_{j=1}^n z_jb_j = 0$, and subtracting this from (2), we arrive at $\sum_{i=1}^m (x_i - w_i)f(a_i) = 0$. Since f is an injective homomorphism, we have $\sum_{i=1}^m (x_i - w_i)a_i = 0$ in A , and so $(x_1 - w_1, \dots, x_m - w_m)$ is a linear combination of the rows of U . This concludes the proof. \square

3 Topological preliminaries

In this part we summarize notions and results from the literature. They are mostly standard in homotopy theory and can be found in textbooks—see, e.g., Hatcher [13] for topological notions and May [18] for simplicial notions (we also refer to Steenrod [37] as an excellent background text, although its terminology differs somewhat from the more modern usage). However, they are perhaps not widely known to non-topologists, and they are somewhat scattered in the literature. We also aim at conveying some simple intuition behind the various notions and concepts, which is not always easy to get from the literature.

On the other hand, in order to follow the arguments in this paper, for some of the notions it is sufficient to know some properties, and the actual definition is never used directly. Such definitions are usually omitted; instead, we illustrate the notions with simple examples or with an informal explanation.

Even readers with a strong topological background may want to skim this part because of the notation. Moreover, in Section 3.3 we discuss an algorithmic result on the construction of Postnikov systems, which may not be well known.

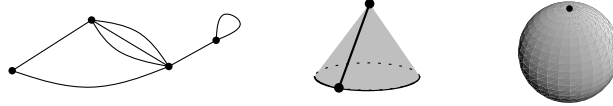
CW-complexes. Below we will state various topological results. Usually they hold for fairly general topological spaces, but not for all topological spaces. The appropriate level of generality for such results is the class of *CW-complexes* (or sometimes spaces homotopy equivalent to CW-complexes).

A reader not familiar with CW-complexes may either look up the definition (e.g., in [13]), or take this just to mean “topological spaces of a fairly general kind, including all simplicial complexes and simplicial sets”. It is also good to know that, similar to simplicial complexes, CW-complexes are made of pieces (*cells*) of various dimensions, where the 0-dimensional cells are also called *vertices*. There is only one place, in Section 4.1, where a difference between CW-complexes and simplicial sets becomes somewhat important, and there we will stress this.

3.1 Simplicial sets

Simplicial sets are our basic device for representing topological spaces and their maps in our algorithm. Here we introduce them briefly, with emphasis on the ideas and intuition, referring to Friedman [8] for a very friendly thorough introduction, to [3, 18] for older compact sources, and to [10] for a more modern and comprehensive treatment.

A *simplicial set* can be thought of as a generalization of simplicial complexes. Similar to a simplicial complex, a simplicial set is a space built of vertices, edges, triangles, and higher-dimensional simplices, but simplices are allowed to be glued to each other and to themselves in more general ways. For example, one may have several 1-dimensional simplices connecting the same pair of vertices, a 1-simplex forming a loop, two edges of a 2-simplex identified to create a cone, or the boundary of a 2-simplex all contracted to a single vertex, forming an S^2 .



However, unlike for the still more general *CW-complexes*, a simplicial set can be described purely combinatorially.

Another new feature of a simplicial set, in comparison with a simplicial complex, is the presence of *degenerate simplices*. For example, the edges of the triangle with a contracted boundary (in the last example above) do not disappear—formally, each of them keeps a phantom-like existence of a degenerate 1-simplex.

Simplices, face and degeneracy operators. A simplicial set X is represented as a sequence (X_0, X_1, X_2, \dots) of mutually disjoint sets, where the elements of X_m are called the *m-simplices* of X . For every $m \geq 1$, there are $m + 1$ mappings $\partial_0, \dots, \partial_m: X_m \rightarrow X_{m-1}$ called *face operators*; the meaning is that for a simplex $\sigma \in X_m$, $\partial_i \sigma$ is the face of σ obtained by deleting the i th vertex. Moreover, there are $m + 1$ mappings $s_0, \dots, s_m: X_m \rightarrow X_{m+1}$ (opposite direction) called the *degeneracy operators*; the meaning of $s_i \sigma$ is the degenerate simplex obtained from σ by duplicating the i th vertex. A simplex is called *degenerate* if it lies in the image of some s_i ; otherwise, it is *nondegenerate*. There are natural axioms that the ∂_i and the s_i have to satisfy, but we will not list them here, since we won’t really use them (and the usual definition of a simplicial set is formally different anyway, expressed in the language of category theory).

Examples. Here we sketch some basic examples of simplicial sets; again, we won’t provide all details, referring to [8]. Let Δ^n denote the standard n -dimensional simplex regarded as a simplicial set. For $n = 0$, $(\Delta^0)_m$ consists of a single simplex, denoted by 0^m , for every $m = 0, 1, \dots$; 0^0 is the only nondegenerate simplex. The face and degeneracy operators are defined in the only possible way.

For $n = 1$, Δ^1 has two 0-simplices (vertices), say 0 and 1, and in general there are $m + 2$ simplices in $(\Delta^1)_m$; we can think of the i th one as containing i copies of the vertex 0 and $m + 1 - i$ copies of the vertex 1, $i = 0, 2, \dots, m + 1$. For n arbitrary, the m -simplices of Δ^n can be thought of as all nondecreasing $(m + 1)$ -term sequences with entries in $\{0, 1, \dots, n\}$; the ones with all terms distinct are nondegenerate.

In a similar fashion, every simplicial complex K can be converted into a simplicial set X in a canonical way; however, first we need to fix a linear ordering of the vertices. The nondegenerate m -simplices of X are in one-to-one correspondence with the m -simplices of K , but many degenerate simplices show up as well.

Finally we mention a “very infinite” but extremely instructive example, the *singular set*, which contributed significantly to the invention of simplicial sets—as Steenrod [37] puts it, the definition of a simplicial set is obtained by writing down fairly obvious properties of the singular set. For a topological space Y , the singular set $S(Y)$ is the simplicial set whose m -simplices are all continuous maps of the standard m -simplex into Y . The i th face operator $\partial_i: S(Y)_m \rightarrow S(Y)_{m-1}$ is given by the composition with a canonical mapping that sends the standard $(m - 1)$ -simplex to the i th face of the standard m -simplex. Similarly, the i th degeneracy operator is induced by the canonical mapping that collapses the standard $(m + 1)$ -simplex to its i th m -dimensional face and then identifies this face with the standard m -simplex, preserving the order of the vertices.

Geometric realization. Similar to a simplicial complex, each simplicial set X defines a topological space $|X|$ (the *geometric realization of X*), uniquely up to homeomorphism. Intuitively, one takes disjoint geometric simplices corresponding to the nondegenerate simplices of X , and glues them together according to the identifications implied by the face and degeneracy operators (we again refer to the literature, especially to [8], for a formal definition).

We note that the degenerate simplices play no role in the geometric realization (and, in a computer representation, they can also be included only implicitly). Their usefulness lies mainly in allowing for a uniform and compact formulation of some definitions and operations.

k -reduced simplicial sets. A simplicial set X is called *k -reduced* if it has a single vertex and no nondegenerate simplices in dimensions 1 through k . Such an X is necessarily k -connected.

A similar terminology can also be used for CW-complexes; k -reduced means a single vertex (0-cell) and no cells in dimensions 1 through k .

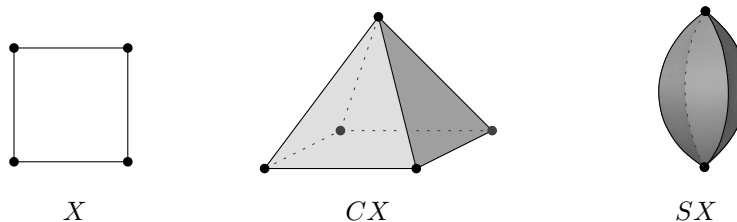
Products. The *product* $X \times Y$ of two simplicial sets is formally defined in an incredibly simple way: we have $(X \times Y)_m := X_m \times Y_m$ for every m , and the face and degeneracy operators work componentwise; e.g., $\partial_i(\sigma, \tau) := (\partial_i\sigma, \partial_i\tau)$. As expected, the product of simplicial sets corresponds to the Cartesian product of the geometric realizations, i.e., $|X \times Y| \cong |X| \times |Y|$.¹³ The simple definition hides some intricacies, though, as one can guess after observing that, for example, the product of two 1-simplices is not a simplex—so the above definition has to imply some canonical way of triangulating the product. It indeed does, and here the degenerate simplices deserve their bread.

Cone and suspension. Given a simplicial set X , the *cone* CX is a simplicial set obtained by adding a new vertex $*$ to X , taking all simplices of X , and, for every m -simplex $\sigma \in X_m$ and every $i \geq 1$, adding to CX the $(m + i)$ -simplex obtained from σ by adding i copies of $*$.

¹³To be precise, the product of topological spaces on the right-hand side should be taken in the category of k -spaces; but for the spaces we encounter, it is the same as the usual product of topological spaces.

In particular, the nondegenerate simplices of CX are the nondegenerate simplices of X plus the cones over these (obtained by adding a single copy of $*$). We skip the definition of face and degeneracy operators for CX as usual. The definitions are discussed, e.g., in [10, Chapter III.5], although there they are given in a more abstract language, and later (in Section 5.3 below) we will state the concrete properties of CX that we will need.

We will also need the *suspension* SX ; this is the simplicial set CX/X obtained from CX by contracting all simplices of X into a single vertex. The following picture illustrates both of the constructions for a 1-dimensional X :



Topologically, SX is the usual (unreduced) suspension of X , which is often presented as erecting a double cone over X (or a join with an S^0). This would also be the “natural” way of defining the suspension for a simplicial complex, but the above definition for simplicial sets is combinatorially different, although topologically equivalent. Even if X is a simplicial complex, SX is not. For us, the main advantage is that the simplicial structure of SX is particularly simple; namely, for $m > 0$, the m -simplices of SX are in one-to-one correspondence with the $(m - 1)$ -simplices of X .¹⁴

Simplicial maps and homotopies. Simplicial sets serve as a combinatorial way of describing a topological space; in a similar way, simplicial maps provide a combinatorial description of continuous maps.

A *simplicial map* $f: X \rightarrow Y$ of simplicial sets X, Y consists of maps $f_m: X_m \rightarrow Y_m$, $m = 0, 1, \dots$, that commute with the face and degeneracy operators. We denote the set of all simplicial maps $X \rightarrow Y$ by $\text{SMap}(X, Y)$.¹⁵

A simplicial map $f: X \rightarrow Y$ induces a continuous map $|f|: |X| \rightarrow |Y|$ of the geometric realizations in a natural way (we again omit the precise definition). Often we will take the usual liberty of omitting $|\cdot|$ and not distinguishing between simplicial sets and maps and their geometric realizations.

¹⁴Let us also remark that in homotopy-theoretic literature, one often works with *reduced* cone and suspension, which are appropriate for the category of pointed spaces and maps. For example, the *reduced suspension* ΣX is obtained from SX by collapsing the segment that connects the apex of CX to the basepoint of X . For CW-complexes, ΣX and SX are homotopy equivalent, so the difference is insignificant for our purposes.

¹⁵There is a technical issue to be clarified here, concerning *pointed maps*. We recall that a *pointed space* (X, x_0) is a topological space X with a choice of a distinguished point $x_0 \in X$ (the *basepoint*). In a CW-complex or simplicial set, we will always assume the basepoint to be a vertex. A *pointed map* $(X, x_0) \rightarrow (Y, y_0)$ of pointed spaces is a continuous map sending x_0 to y_0 . Homotopies of pointed maps are also meant to be pointed; i.e., they must keep the image of the basepoint fixed. The reader may recall that, for example, the homotopy groups $\pi_k(Y)$ are really defined as homotopy classes of pointed maps.

If X, Y are simplicial sets, X is arbitrary, and Y is a 1-reduced (thus, it has a single vertex, which is the basepoint), as will be the case for the targets of simplicial maps in our algorithm, then every simplicial map is automatically pointed. Thus, in this case, we need not worry about pointedness.

A topological counterpart of this is that, if Y is a 1-connected CW-complex, then every map $X \rightarrow Y$ is (canonically) homotopic to a map sending x_0 to y_0 , and thus $[X, Y]$ is canonically isomorphic to the set of all homotopy classes of pointed maps $X \rightarrow Y$.

Of course, not all continuous maps are induced by simplicial maps. But the usefulness of simplicial sets for our algorithm (and many other applications) stems mainly from the fact that, if the target Y has the *Kan extension property*, then *every* continuous map $\varphi: |X| \rightarrow |Y|$ is *homotopic* to a simplicial map $f: X \rightarrow Y$.¹⁶

The *Kan extension property* is a certain property of a simplicial set (and the simplicial sets having it are called *Kan simplicial sets*), which need not be spelled out here—it will suffice to refer to standard results to check the property where needed. In particular, every *simplicial group* is a Kan simplicial set, where a simplicial group G is a simplicial set for which every G_m is endowed with a group structure, and the face and degeneracy operators are group homomorphisms (we will see examples in Section 3.2 below).

Homotopies of simplicial maps into a Kan simplicial set can also be represented simplicially. Concretely, a *simplicial homotopy* between two simplicial maps $f, g: X \rightarrow Y$ is a simplicial map $F: X \times \Delta^1 \rightarrow Y$ such that $F|_{X \times \{0\}} = f$ and $F|_{X \times \{1\}} = g$; here, as we recall, Δ^1 represents the geometric 1-simplex (segment) as a simplicial set, and, with some abuse of notation, $\{0\}$ and $\{1\}$ are the simplicial subsets of Δ^1 representing the two vertices. Again, if Y is a Kan simplicial set, then two simplicial maps f, g into Y are simplicially homotopic *iff* they are homotopic in the usual sense as continuous maps.

Not surprisingly, there is a price to pay for the convenience of representing all continuous maps and homotopies simplicially: a Kan simplicial set Y necessarily has infinitely many simplices in every dimension (except for some trivial cases), and thus we need nontrivial techniques for representing it in a computer. Fortunately, the relevant Y 's in our case have a sufficiently regular structure and can be handled; suitable techniques have been developed and presented in [31, 24, 23, 25, 27].

3.2 Eilenberg–MacLane spaces and cohomology

Cohomology. We will need some terminology from (simplicial) cohomology, such as cochains, cocycles, and cohomology groups. However, these will be mostly a convenient bookkeeping device for us, and we won't need almost any properties of cohomology.

For a simplicial complex X , an integer $n \geq 0$, and an Abelian group π , an *n -dimensional cochain* with values in π is an arbitrary mapping $c^n: X_n \rightarrow \pi$, i.e., a labeling of the n -dimensional simplices of X with elements of π . The set of all n -dimensional cochains is (traditionally) denoted by $C^n(X; \pi)$; with componentwise addition, it forms an Abelian group.

For a simplicial *set* X , we define $C^n(X; \pi)$ to consist only of cochains in which all *degenerate* simplices receive value 0 (these are sometimes called *normalized cochains*).

Given an n -cochain c^n , the *coboundary* of c^n is the $(n+1)$ -cochain $d^{n+1} = \delta c^n$ whose value on a $\tau \in X_{n+1}$ is the sum of the values of c^n over the n -faces of τ (taking orientations into account); formally,

$$d^{n+1}(\tau) = \sum_{i=0}^{n+1} (-1)^i c^n(\partial_i \tau).$$

A cochain c^n is a *cocycle* if $\delta c^n = 0$; $Z^n(X; \pi) \subseteq C^n(X; \pi)$ is the subgroup of all cocycles (Z for *koZyklus*), i.e., the kernel of δ . The subgroup $B^n(X; \pi) \subseteq C^n(X; \pi)$ of all *coboundaries*

¹⁶The reader may be familiar with the *simplicial approximation theorem*, which states that for every continuous map $\varphi: |K| \rightarrow |L|$ between the polyhedra of simplicial complexes, there is a simplicial map of a *sufficiently fine subdivision* of K into L that is homotopic to φ . The crucial difference is that in the case of simplicial sets, if Y has the Kan extension property, we need not subdivide X at all!

is the image of δ ; that is, c^n is a coboundary if $c^n = \delta b^{n-1}$ for some $(n-1)$ -cochain b^{n-1} .

The n th (simplicial) *cohomology group* of X is the factor group

$$H^n(X; \pi) := Z^n(X; \pi) / B^n(X; \pi)$$

(for this to make sense, of course, one needs the basic fact $\delta \circ \delta = 0$).

Eilenberg–MacLane spaces as “simple ranges”. The homotopy groups $\pi_k(Y)$ are among the most important invariants of a topological space Y . The group $\pi_k(Y)$ collects information about the “ k -dimensional structure” of Y by probing Y with all possible maps from S^k . Here the sphere S^k plays a role of the “simplest nontrivial” k -dimensional space; indeed, in some respects, for example concerning homology groups, it is as simple as one can possibly get.

However, as was first revealed by the famous *Hopf map* $S^3 \rightarrow S^2$, the spheres are not at all simple concerning maps going *into* them. In particular, the groups $\pi_k(S^n)$ are complicated and far from understood, in spite of a huge body of research devoted to them. So if one wants to probe a space X with maps going *into* some “simple nontrivial” space, then spaces other than spheres are needed—and the Eilenberg–MacLane spaces can play this role successfully.

Given an Abelian group π and an integer $n \geq 1$, an *Eilenberg–MacLane space* $K(\pi, n)$ is defined as any topological space T with $\pi_n(T) \cong \pi$ and $\pi_k(T) = 0$ for all $k \neq n$. It is not difficult to show that a $K(\pi, n)$ exists (by taking a wedge of n -spheres and inductively attaching balls of dimensions $n+1, n+2, \dots$ to kill elements of the various homotopy groups), and it also turns out that $K(\pi, n)$ is unique up to homotopy equivalence.¹⁷

The circle S^1 is (one of the incarnations of) a $K(\mathbb{Z}, 1)$, and $K(\mathbb{Z}_2, 1)$ can be represented as the infinite-dimensional real projective space, but generally speaking, the spaces $K(\pi, n)$ do not look exactly like very simple objects.

Maps into $K(\pi, n)$. Yet the following elegant fact shows that the $K(\pi, n)$ indeed constitute “simple” targets of maps.

Lemma 3.1. *For every $n \geq 1$ and every Abelian group π , we have*

$$[X, K(\pi, n)] \cong H^n(X; \pi),$$

where X is a simplicial complex (or a CW-complex).

This is a basic and standard result (e.g., [18, Lemma 24.4] in a simplicial setting), but nevertheless we will sketch an intuitive geometric proof, since it explains why maps into $K(\pi, n)$ can be represented discretely, by cocycles, and this is a key step towards representing maps in our algorithm.

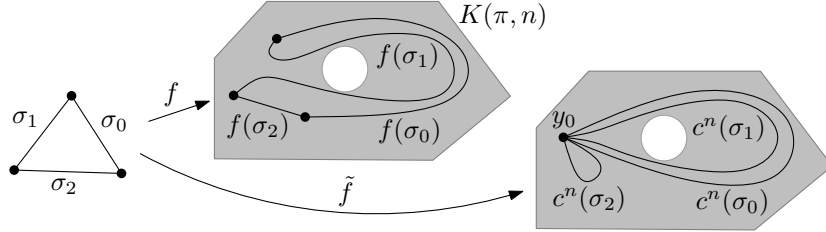
Sketch of proof. For simplicity, let X be a finite simplicial complex (the argument works for a CW-complex in more or less the same way), and let us consider an arbitrary continuous map $f: |X| \rightarrow K(\pi, n)$, $n \geq 2$.

First, let us consider the restriction of f to the $(n-1)$ -skeleton $X^{(n-1)}$ of X . Since by definition, $K(\pi, n)$ is $(n-1)$ -connected, $f|_{X^{(n-1)}}$ is homotopic to the constant map sending $X^{(n-1)}$ to a single point y_0 (we can imagine pulling the images of the simplices to y_0 one by one, starting with vertices, continuing with 1-simplices, etc., up to $(n-1)$ -simplices). Next, the homotopy of $f|_{X^{(n-1)}}$ with this constant map can be extended to a homotopy of f with a

¹⁷Provided that we restrict to spaces that are homotopy equivalent to CW-complexes.

map \tilde{f} defined on all of X (this is a standard fact known as the *homotopy extension property* of X , valid for all CW-complexes, among others). Thus, $\tilde{f} \sim f$ sends $X^{(n-1)}$ to y_0 .

Next, we consider an n -simplex σ of X . All of its boundary now goes to y_0 , and so the restriction of \tilde{f} to σ can be regarded as a map $S^n \rightarrow K(\pi, n)$ (since collapsing the boundary of an n -simplex to a point yields an S^n). Thus, up to homotopy, $\tilde{f}|_\sigma$ is described by an element of $\pi_n(K(\pi, n)) = \pi$. In this way, \tilde{f} defines a cochain $c^n = c^n_{\tilde{f}} \in C^n(X; \pi)$. The following picture captures this schematically:

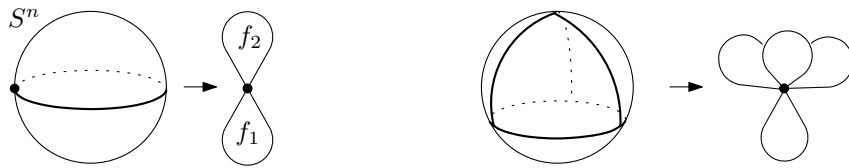


The target space $K(\pi, n)$ is illustrated as having a hole “responsible” for the nontriviality of π_n .

We note that \tilde{f} is not determined uniquely by f , and $c^n_{\tilde{f}}$ may also depend on the choice of \tilde{f} .

Next, we observe that every cochain of the form $c^n_{\tilde{f}}$ is actually a *cocycle*. To this end, we consider an $(n+1)$ -simplex $\tau \in X_{n+1}$. Since \tilde{f} is defined on all of τ , the restriction $\tilde{f}|_{\partial\tau}$ to the boundary is nullhomotopic. At the same time, $\tilde{f}|_{\partial\tau}$ can be regarded as the sum of the elements of $\pi_n(K(\pi, n))$ represented by the restrictions of \tilde{f} to the n -dimensional faces of τ .

Indeed, for any space Y the sum $[f]$ of two elements $[f_1], [f_2] \in \pi_n(Y)$ can be represented by contracting an $(n-1)$ -dimensional “equator” of S^n to the basepoint, thus obtaining a wedge of two S^n ’s, and then defining f to be f_1 on one of these and f_2 on the other, as indicated in the picture below on the left (this time for $n=2$). Similarly, in our case, the sum of the maps on the facets of τ can be represented by contracting the $(n-1)$ -skeleton of τ to a point, and thus obtaining a wedge of $n+2$ n -spheres.



Therefore, we have $(\delta c^n)(\tau) = 0$, and $c^n = c^n_{\tilde{f}} \in Z^n(X; \pi)$ as claimed.

Conversely, given any $z^n \in Z^n(X; \pi)$, one can exhibit a map $\tilde{f}: X \rightarrow K(\pi, n)$ with $c^n_{\tilde{f}} = z^n$. Such an \tilde{f} is built one simplex of X at a time. First, all simplices of dimension at most $n-1$ are sent to y_0 . For every $\sigma \in X_n$, we choose a representative of the element $z^n(\sigma) \in \pi_n(K(\pi, n))$, which is a (pointed) map $S^n \rightarrow K(\pi, n)$, and use it to map σ . Then for $\tau \in X_{n+1}$, \tilde{f} can be extended to τ , since $\tilde{f}|_{\partial\tau}$ is nullhomotopic by the cocycle condition for z^n . Finally, for a simplex ω of dimension larger than $n+1$, the \tilde{f} constructed so far is necessarily nullhomotopic on $\partial\omega$ because $\pi_k(K(\pi, n)) = 0$ for all $k > n$, and thus an extension to ω is always possible.

We hope that this may convey some idea where the cocycle representation of maps into $K(\pi, n)$ comes from. By similar, but a little more complicated considerations, which we omit

here, one can convince oneself that two maps $f, g: X \rightarrow K(\pi, n)$ are homotopic exactly when the corresponding cocycles c_f^n and c_g^n differ by a coboundary. In particular, for a given f , the cocycle c_f^n may depend on the choice of \tilde{f} , but the cohomology class $c_f^n + B^n(X; \pi)$ doesn't. This finishes the proof sketch. \square

A Kan simplicial model of $K(\pi, n)$. The Eilenberg–MacLane spaces $K(\pi, n)$ can be represented as Kan simplicial sets, and actually as simplicial groups, in an essentially unique way; we will keep the notation $K(\pi, n)$ for this simplicial set as well.

Namely, the set of m -simplices of $K(\pi, n)$ is given by the amazing formula

$$K(\pi, n)_m := Z^n(\Delta^m; \pi).$$

More explicitly, an m -simplex σ can be regarded as a labeling of the n -dimensional faces of the standard m -simplex by elements of the group π ; moreover, the labels must add up to 0 on every $(n+1)$ -face. There are $\binom{m+1}{n+1}$ nondegenerate n -faces of Δ^m , and so an m -simplex $\sigma \in K(\pi, n)_m$ is determined by an ordered $\binom{m+1}{n+1}$ -tuple of elements of π .

It is not hard to define the face and degeneracy operators for $K(\pi, n)$, but we omit this since we won't use them explicitly (see, e.g., [18, 27]). It suffices to say that the *degenerate* σ are precisely those labellings with two facets of Δ^m labelled identically and zero everywhere else.

In particular, for every $m \geq 0$, we have an m -simplex in $K(\pi, n)$ formed by the zero n -cochain, which is nondegenerate for $m = 0$ and degenerate for $m > 0$, and which we write simply as 0 (with the dimension understood from context). It is remarkable that the zero n -cochain on Δ^0 is the only vertex of the simplicial set $K(\pi, n)$ for $n > 0$.

We won't prove that this is indeed a simplicial model of $K(\pi, n)$. Let us just note that $K(\pi, n)$ is $(n-1)$ -reduced, and its n -simplices correspond to elements of π (since an n -cocycle on Δ^n is a labeling of the single nondegenerate n -simplex of Δ^n by an element of π). Thus, each n -simplex of $K(\pi, n)$ “embodies” one of the possible ways of mapping the interior of Δ^n into $K(\pi, n)$, given that the boundary goes to the basepoint. The $(n+1)$ -simplices then “serve” to get the appropriate addition relations among the just mentioned maps, so that this addition works as that in π , and the higher-dimensional simplices kill all the higher homotopy groups.

The (elementwise) addition of cochains makes $K(\pi, n)$ into a simplicial group, and consequently, $K(\pi, n)$ is a Kan simplicial set.

The simplicial sets $E(\pi, n)$. The m -simplices in the simplicial Eilenberg–MacLane spaces as above are all n -cocycles on Δ^m . If we take all n -cochains, we obtain another simplicial set called $E(\pi, n)$. Thus, explicitly,

$$E(\pi, n)_m := C^n(\Delta^m; \pi).$$

As a topological space, $E(\pi, n)$ is contractible, and thus not particularly interesting topologically in itself, but it makes a useful companion to $K(\pi, n)$. Obviously, $K(\pi, n) \subseteq E(\pi, n)$, but there are also other, less obvious relationships.

Since an m -simplex $\sigma \in E(\pi, n)$ is formally an n -cochain, we can take its coboundary $\delta\sigma$. This is an $(n+1)$ -coboundary (and thus also cocycle), which we can interpret as an m -simplex of $K(\pi, n+1)$. It turns out that this induces a *simplicial* map $E(\pi, n) \rightarrow K(\pi, n+1)$, which

is (with the usual abuse of notation) also denoted by δ . This map is actually surjective, since the relevant cohomology groups of Δ^m are all zero and thus all cocycles are also coboundaries.

Simplicial maps into $K(\pi, n)$ and $E(\pi, n)$. We have the following “simplicial” counterpart of Lemma 3.1:

Lemma 3.2. *For every simplicial complex (or simplicial set) X , we have*

$$\mathrm{SMap}(X, K(\pi, n)) \cong Z^n(X; \pi) \text{ and } \mathrm{SMap}(X, E(\pi, n)) \cong C^n(X; \pi).$$

We refer to [18, Lemma 24.3] for a proof; here we just describe how the isomorphism¹⁸ works, i.e., how one passes between cochains and simplicial maps. This is not hard to guess from the formal definition—there is just one way to make things match formally.

Namely, given a $c^n \in C^n(X; \pi)$, we want to construct the corresponding simplicial map $s = s(c^n): X \rightarrow E(\pi, n)$. We consider an m -simplex $\sigma \in X_m$. There is exactly one way of inserting the standard m -simplex Δ^m to the “place of σ ” into X ; more formally, there is a unique simplicial map $i_\sigma: \Delta^m \rightarrow X$ that sends the m -simplex of Δ^m to σ (indeed, a simplicial map has to respect the ordering of vertices, implicit in the face and degeneracy operators). Thus, for every such σ , the cochain c^n defines a cochain $i_\sigma^*(c^n)$ on Δ^m (the labels of the n -faces of σ are pulled back to Δ^m), and that cochain is taken as the image $s(\sigma)$.

For the reverse direction, i.e., from a simplicial map s to a cochain, it suffices to look at the images of the n -simplices under s : these are n -simplices of $E(\pi, n)$ which, as we have seen, can be regarded as elements of π —thus, they define the values of the desired n -cochain.

Simplicial homotopy in $\mathrm{SMap}(X, K(\pi, n))$. Now that we have a description of simplicial maps $X \rightarrow K(\pi, n)$, we will also describe homotopies (or equivalently, simplicial homotopies) among them. It turns out that the additive structure (cocycle addition) on $\mathrm{SMap}(X, K(\pi, n)) \cong Z^n(X; \pi)$ reduces the question of whether two maps represented by cocycles c_1 and c_2 are homotopic to the question whether their difference $c_1 - c_2$ is *nullhomotopic* (homotopic to a constant map).

Lemma 3.3. *Let $c_1^n, c_2^n \in Z^n(X; \pi)$ be two cocycles. Then the simplicial maps $s_1, s_2 \in \mathrm{SMap}(X, K(\pi, n))$ represented by c_1^n, c_2^n , respectively, are simplicially homotopic iff c_1 and c_2 are cohomologous, i.e., $c_1 - c_2 \in B^n(X; \pi)$.*

We refer to [18, Theorem 24.4] for a proof. We also remark that a simplicial version of Lemma 3.1 is actually proved using Lemmas 3.2 and 3.3.

3.3 Postnikov systems

Now that we have a combinatorial representation of maps from X into an Eilenberg–MacLane space, and of their homotopies, it would be nice to have a similar thing for other target spaces Y . Expressing Y through its simplicial *Postnikov system* comes as close to fulfilling this plan as seems reasonably possible.

Postnikov systems are somewhat complicated objects, and so we will not discuss them in detail, referring to standard textbooks ([13] in general and [18] for the simplicial case) instead.

¹⁸Both sets carry an Abelian group structure, and the bijection between them preserves these. For the set $Z^n(X; \pi)$ of cocycles, the group structure is given by the usual addition of cocycles. For the set $\mathrm{SMap}(X, K(\pi, n))$ of simplicial maps, the group structure is given by the fact that $K(\pi, n)$ is a *simplicial Abelian group*, so simplicial maps into it can be added componentwise (simplexwise).

First we will explain some features of a Postnikov system in the setting of topological spaces and continuous maps; this part, strictly speaking, is not necessary for the algorithm. Then we introduce a simplicial version of a Postnikov system, and summarize the properties we will actually use. Finally, we will present the subroutine used to compute Postnikov systems.

Postnikov systems on the level of spaces and continuous maps. Let Y be a CW-complex. A *Postnikov system* (also called a *Postnikov tower*) for Y is a sequence of spaces P_0, P_1, P_2, \dots , where P_0 is a single point, together with maps $\varphi_i: Y \rightarrow P_i$ and $p_i: P_i \rightarrow P_{i-1}$ such that $p_i \circ \varphi_i = \varphi_{i-1}$, i.e., the following diagram commutes:

$$\begin{array}{ccc}
 & & \vdots \\
 & & P_2 \\
 & \nearrow \varphi_2 & \downarrow p_2 \\
 & & P_1 \\
 & \nearrow \varphi_1 & \downarrow p_1 \\
 Y & \xrightarrow{\varphi_0} & P_0
 \end{array}$$

Informally, the P_i , called the *stages* of the Postnikov system, can be thought of as successive stages in a process of building Y (or rather, a space homotopy equivalent to Y) “layer by layer” from the Eilenberg–Mac Lane spaces $K(\pi_i(Y), i)$.

More formally, it is required that for each i , the mapping φ_i induces an isomorphism $\pi_j(Y) \cong \pi_j(P_i)$ of homotopy groups for every $j \leq i$, while $\pi_j(P_i) = 0$ for all $j > i$. These properties suffice to define a Postnikov system uniquely up to homotopy equivalence, provided that Y is 0-connected and the P_i are assumed to be CW-complexes; see, e.g., Hatcher [13, Section 4.3].

For the rest of this paper, we will abbreviate $\pi_i(Y)$ to π_i .

One usually works with Postnikov systems with additional favorable properties, sometimes called *standard Postnikov systems*, and for these to exist, more assumptions on Y are needed—in particular, they do exist if Y is 1-connected. In this case, the first two stages, P_0 and P_1 , are trivial, i.e., just one-point spaces.

Standard Postnikov systems on the level of topological spaces are defined using the notion of *principal fibration*, which we don’t need/want to define here. Let us just sketch informally how P_i is built from P_{i-1} and $K(\pi_i, i)$. *Locally*, P_i “looks like” the product $P_{i-1} \times K(\pi_i, i)$, in the sense that the fiber $p_i^{-1}(x)$ of every point $x \in P_{i-1}$ is (homotopy equivalent to) $K(\pi_i, i)$. However, *globally* P_i is usually *not* the product as above; rather, it is “twisted” (technically, it is the total space of the fibration $K(\pi_i, i) \rightarrow P_i \xrightarrow{p_i} P_{i-1}$). A somewhat simple-minded analog is the way the Möbius band is made by putting a segment “over” every point of S^1 , looking locally like the product $S^1 \times [-1, 1]$ but globally, of course, very different from that product.

The way of “twisting” the $K(\pi_i, i)$ over P_{i-1} to form P_i is specified, for reasons that would need a somewhat lengthy explanation, by a mapping $k_{i-1}: P_{i-1} \rightarrow K(\pi_i, i+1)$. As we know, each such map k_{i-1} can be represented by a cocycle in $Z^{i+1}(P_{i-1}; \pi_i)$, and since it really suffices to know k_{i-1} only up to homotopy, it is enough to specify it by an element of the cohomology group $H^{i+1}(P_{i-1}; \pi_i)$. This element is also commonly denoted by k_{i-1} and called the $(i-1)$ st *Postnikov class*¹⁹ of Y .

¹⁹In the literature, *Postnikov factor* or *Postnikov invariant* are also used with the same meaning.

The beauty of the thing is that P_i , which conveys, in a sense, complete information about the homotopy of Y up to dimension i , can be reconstructed from the *discrete* data given by $\pi_2, k_2, \pi_3, k_3, \dots, k_{i-1}, \pi_i$.

For our purposes, a key fact is the following:

Proposition 3.4. *If X is a CW-complex of dimension at most i , and Y is a 1-connected CW-complex, then there is a bijection between $[X, Y]$ and $[X, P_i]$ (which is induced by composition with the map φ_i).*

Thus, for computing $[X, Y]$ in Theorem 1.1, it suffices to compute $[X, P_{2d-2}]$, for the appropriate stage P_{2d-2} of a Postnikov system of Y . This will be done inductively, i.e., the main step of the algorithm will be to compute $[X, P_i]$ from $[X, P_{i-1}]$, $i \leq 2d - 2$.

Simplicial Postnikov systems. To use Postnikov systems algorithmically, we represent the objects by simplicial sets and maps (this was actually the setting in which Postnikov originally defined them). Concretely, we will use the so-called *pullback representation* (as opposed to some other sources, where a *twisted product* representation can be found—but these representations can be converted into one another without much difficulty).

We let $K(\pi, n)$ and $E(\pi, n)$ stand for the particular simplicial sets as in Section 3.2. The i -th stage P_i of the Postnikov system for Y is represented as a simplicial subset of the product $P_{i-1} \times E_i \subseteq E_0 \times E_1 \times \dots \times E_i$, where $E_j := E(\pi_j, j)$. An m -simplex of P_i can thus be written as $(\sigma^0, \dots, \sigma^{i-1}, \sigma^i)$, where $\sigma^j \in C^j(\Delta^m, \pi_j)$ is a simplex of E_j . It will also be convenient to write $(\sigma^0, \dots, \sigma^{i-1}) \in P_{i-1}$ as σ and thus write a simplex of P_i in the form (σ, σ^i) .

We will introduce the following convenient abbreviations for the Eilenberg–MacLane spaces appearing in the Postnikov system (the first of them is quite standard):

$$\begin{aligned} K_{i+1} &:= K(\pi_i, i+1), \\ L_i &:= K(\pi_i, i). \end{aligned}$$

The simplicial version of (a representative of) the Postnikov class k_{i-1} is a simplicial map

$$k_{i-1} \in \text{SMap}(P_{i-1}, K_{i+1}).$$

Since K_{i+1} is an Eilenberg–MacLane space, we can, and will, also represent k_{i-1} as a cocycle in $Z^{i+1}(P_{i-1}, \pi_i)$.

In this version, instead of “twisting”, k_{i-1} is used to “cut out” P_i from the product $P_{i-1} \times E_i$, as follows:

$$P_i := \{(\sigma, \sigma^i) \in P_{i-1} \times E_i : k_{i-1}(\sigma) = \delta \sigma^i\}, \quad (3)$$

where $\delta: E_i \rightarrow K_{i+1}$ is given by the coboundary operator, as was described above after the definition of $E(\pi, n)$. The map $p_i: P_i \rightarrow P_{i-1}$ in this setting is simply the projection forgetting the last coordinate, and so it need not be specified explicitly.

We remark that this describes what the simplicial Postnikov system looks like, but it doesn’t say when it really is a Postnikov system for Y . We won’t discuss the appropriate conditions here; we will just accept a guarantee of the algorithm in Theorem 3.5 below, that it computes a valid Postnikov system for Y , and in particular, such that it fulfills Proposition 3.4.

We also state another important property of the stages P_i of the simplicial Postnikov system of a simply connected Y : they are Kan simplicial sets (see, e.g. [2]). Thus, for any

simplicial set X , there is a bijection between the set of simplicial maps $X \rightarrow P_i$ modulo simplicial homotopy and the set of homotopy classes of continuous maps between the geometric realizations. Slightly abusing notation, we will denote both sets by $[X, P_i]$.

On computing Postnikov systems. For our purposes, we shall say that a (1-connected) simplicial set Y has a *locally effective Postnikov system with n stages* if the following are available:

- The homotopy groups π_2, \dots, π_n (provided with a fully effective representation).²⁰
- (k_{i-1} locally effective) An algorithm that, given an $(i+1)$ -simplex $\sigma \in P_{i-1}$, returns the value of the cocycle k_{i-1} on σ , i.e., $k_{i-1}(\sigma) \in \pi_i$, $i \leq n$.²¹
- (φ_i locally effective) An algorithm that, given a simplex σ of Y and an index $i \leq n$, computes $\varphi_i(\sigma) \in P_i$. (Actually, $i = n$ suffices; the other φ_i follow by projection. Moreover, using the cochain representation of maps into P_i , it suffices to have an algorithm working only with simplices σ of dimension at most n .)

In the case with π_2 through π_n all finite, each P_i , $i \leq n$, has finitely many simplices in the relevant dimensions, and so a locally effective Postnikov system can be represented simply by a lookup table. Brown [2] gave an algorithm for computing a simplicial Postnikov system in this (restricted) setting.

The methods of *effective homology*, as explained, e.g., in [27], combined with the construction of a Postnikov system as given, e.g., in Spanier [35, Section 8.3] (in particular, Corollary 7 there), lead to the following result.

Theorem 3.5. *Let Y be a 1-connected simplicial set that has finitely many nondegenerate simplices (e.g., as obtained from a finite simplicial complex), or more generally, that is equipped with effective homology in the sense of [27]. Then, for every n , a locally effective Postnikov system for Y with n stages can be constructed.*

We will not specify the notion of effective homology here, and neither some of the notions in the following comments, referring to [27]. As for effective homology, it suffices to say that this means enhancing the representation of Y with an auxiliary computational structure, which allows one, first, to get a fully effective representation of each homology group of Y , and second, to perform various topological operations with Y (e.g., product, loop space, suspension, etc.) in such a way that the resulting object again comes with effective homology.

More concretely, in the proof of Theorem 3.5, the following operations and constructions play the main role:

- Constructing the total space of a fibration; see [27, Section 8]. Here we consider a fibration $F \rightarrow E \rightarrow B$ of simplicial sets. The input data are the fiber F and the base space B , both with effective homology, an action of a simplicial group G on F , and a

²⁰For our algorithm, it suffices to have the π_i represented as abstract Abelian groups, with no meaning attached to the elements. However, if we ever wanted to translate the elements of $[X, P_i]$ to actual maps $X \rightarrow Y$, we would need the generators of each π_i represented as actual mappings, say simplicial, $S^i \rightarrow Y$.

²¹Let us remark that, by unwrapping the definition, we get that the input $\sigma \in P_{i-1}$ for k_{i-1} means a labeling of the faces of Δ^{i+1} of all dimensions up to $i-1$, where j -faces are labeled by elements of π_j . Readers familiar with obstruction theory may see some formal similarity here: the $(i-1)$ st obstruction determines extendability of a map defined on the i -skeleton to the $(i+1)$ -skeleton, after possibly modifying the map on the interiors of the i -simplices.

locally effective twisting function $\tau: B \rightarrow G$, specifying the total space E as the twisted product $F \times_\tau B$. Then effective homology of E can be computed.

- Constructing the classifying space BG of a simplicial group G . Here G is 0-reduced and given with effective homology, and effective homology of the classifying space BG is computed. In [27, Section 9], an effective homology construction of the loop space ΩX from X is detailed. The dual construction of BG from G is essentially the same, we just “reverse all the arrows”; see [27, Theorem 150].
- A representation with effective homology of the Eilenberg–MacLane space $K(\pi, 1)$ for a finitely generated Abelian group π . This follows Eilenberg and Mac Lane [6, Chap. III]. Effective homology for $K(\pi, n)$ with $n \geq 2$ is then obtained using the recursive relation $K(\pi, n) = B(K(\pi, n - 1))$.

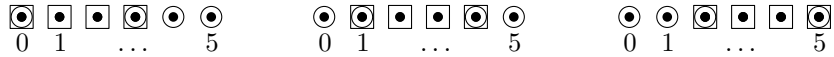
This concludes a sketch of the main ideas in the proof of Theorem 3.5. As was mentioned in the introduction, we plan to elaborate on this topic in a separate paper. In particular, it seems that, concerning running time, the critical part is the base case $n = 1$ in representing $K(\pi, n)$; all the other operations should preserve polynomiality.

An example: the Steenrod square Sq^2 . The Postnikov factors k_i are not at all simple to describe explicitly, even for very simple spaces. As an illustration, we present an example, essentially following [36], where an explicit description is available: this is for $Y = S^d$, $d \geq 3$, and it concerns the first k_i of interest, namely, k_d . It corresponds to the *Steenrod square* Sq^2 in cohomology, which Steenrod [36] invented for the purpose of classifying all maps from a $(d + 1)$ -dimensional complex K into S^d —a special case of the problem treated in our paper.

For concreteness, let us take $d = 3$. Then k_3 receives as the input a labeling of the 3-faces of Δ^5 by elements of $\pi_3(S^3)$, i.e., integers (the lower-dimensional faces are labeled with 0s since $\pi_j(S^3) = 0$ for $j \leq 2$), and it should return an element of $\pi_4(S^3) \cong \mathbb{Z}_2$. Combinatorially, we can thus think of the input as a function $c: \binom{\{0,1,\dots,5\}}{4} \rightarrow \mathbb{Z}$, and the value of k_3 turns out to be

$$\sum_{\sigma, \tau} c(\sigma)c(\tau) \pmod{2},$$

where the sum is over three pairs of 4-tuples σ, τ as indicated in the following picture (σ consists of the circled points and τ of the points marked by squares—there is always a two-point overlap):



This illustrates the nonlinearity of the Postnikov classes.

4 Defining and implementing the group operation on $[X, P_i]$

We recall that the device that allows us to handle the generally infinite set $[X, Y]$ of homotopy classes of maps, under the dimension/connectedness assumption of Theorem 1.1, is an Abelian group structure. We will actually use the group structure on the sets $[X, P_i]$, $d \leq i \leq 2d - 2$. These will be computed inductively, starting with $i = d$ (this is the first nontrivial one).

Such a group structure with good properties exists, and is determined uniquely, because P_i may have nonzero homotopy groups only in dimensions d through $2d - 2$; these are standard topological considerations, which we will review in Section 4.1 below.

However, we will need to work with the underlying binary operation \boxplus_{i*} on the level of representatives, i.e., simplicial maps in $\text{SMap}(X, P_i)$. This operation lacks some of the pleasant properties of a group—e.g., it may fail to be associative. Here considerable care and attention to detail seem to be needed, and for an algorithmic implementation, we also need to use the *Eilenberg–Zilber reduction*, a tool related to the methods of effective homology.

4.1 An H -group structure on a space

H -groups. Let P be a CW-complex. We will consider a *binary operation* on P as a *continuous map* $\mu: P \times P \rightarrow P$. For now, we will stick to writing $\mu(p, q)$ for the result of applying μ to p and q ; later on, we will call the operation \boxplus (with a subscript, actually) and write it in the more usual way as $p \boxplus q$.

The idea of H -groups is that the binary operation μ satisfies the usual group axioms but only *up to homotopy*. To formulate the existence of an inverse in this setting, we will also need an explicit mapping $\nu: P \rightarrow P$, continuous of course, representing *inverse up to homotopy*.

We thus say that

- (HA) μ is *homotopy associative* if the two maps $P \times P \times P \rightarrow P$ given by $(p, q, r) \mapsto \mu(p, \mu(q, r))$ and by $(p, q, r) \mapsto \mu(\mu(p, q), r)$ are homotopic;
- (HN) a distinguished element $o \in P$ (basepoint, assumed to be a vertex in the simplicial set representation) is a *homotopy neutral element* if the maps $P \rightarrow P$ given by $p \mapsto \mu(o, p)$ and $p \mapsto \mu(p, o)$ are both homotopic to the identity id_P ;
- (HI) ν is a *homotopy inverse* if the maps $p \mapsto \mu(\nu(p), p)$ and $p \mapsto \mu(p, \nu(p))$ are both homotopic to the constant map $p \mapsto o$;
- (HC) μ is *homotopy commutative* if μ is homotopic to μ' given by $\mu'(p, q) := \mu(q, p)$.

An *Abelian H -group* thus consists of P, o, μ, ν as above satisfying (HA), (HN), (HI), and (HC).

Of course, every Abelian topological group is also an Abelian H -group. A basic example of an H -group that is typically not a group is the *loop space* ΩY of a topological space Y (see, e.g. [13, Section 4.3]). For readers familiar with the definition of the fundamental group $\pi_1(Y)$, it suffices to say that ΩY is like the fundamental group but *without* factoring the loops according to homotopy.

We also define an *H -homomorphism* of an H -group (P_1, o_1, μ_1, ν_1) into an H -group (P_2, o_2, μ_2, ν_2) in a natural way, as a continuous map $h: P_1 \rightarrow P_2$ with $h(o_1) = o_2$ and such that the two maps $(x, y) \mapsto h(\mu_1(x, y))$ and $(x, y) \mapsto \mu_2(h(x), h(y))$ are homotopic.

A group structure on homotopy classes of maps. For us, an H -group structure on P is a device for obtaining a group structure on the set $[X, P]$ of homotopy classes of maps. In a similar vein, an H -homomorphism $P_1 \rightarrow P_2$ yields a group homomorphism $[X, P_1] \rightarrow [X, P_2]$. Here is a more explicit statement:

Fact 4.1. *Let (P, o, μ, ν) be an Abelian H -group, and let X be a space. Let μ_*, ν_* be the operations defined on continuous maps $X \rightarrow P$ by pointwise composition with μ, ν , respectively*

(i.e., $\mu_*(f, g)(x) := \mu(f(x), g(x))$, $\nu_*(f)(x) := \nu(f(x))$). Then μ_* , ν_* define an Abelian group structure on the set of homotopy classes $[X, P]$ by $[f] + [g] := [\mu_*(f, g)]$ and $-[f] := [\nu_*(f)]$ (with the zero element given by the homotopy class of the map sending all of X to o).

If $h: P_1 \rightarrow P_2$ is an H -homomorphism of Abelian H -groups (P_1, o_1, μ_1, ν_1) and (P_2, o_2, μ_2, ν_2) , then the corresponding map h_* , sending a continuous map $f: X \rightarrow P_1$ to $h_*(f): X \rightarrow P_2$ given by $h_*(f)(x) := h(f(x))$, induces a homomorphism $[h_*]: [X, P_1] \rightarrow [X, P_2]$ of Abelian groups.

This fact is standard, and also entirely routine to prove. We will actually work mostly with a simplicial counterpart (which is proved in exactly the same way, replacing topological notions with simplicial ones everywhere). Namely, if X is a simplicial set, P is a Kan simplicial set, and μ, ν are simplicial maps, then by a composition as above, we obtain maps $\mu_*: \text{SMap}(X, P) \times \text{SMap}(X, P) \rightarrow \text{SMap}(X, P)$ and $\nu_*: \text{SMap}(X, P) \rightarrow \text{SMap}(X, P)$, which induce an Abelian group structure on the set $[X, P]$ of simplicial homotopy classes. Similarly, if $h: P_1 \rightarrow P_2$ is a simplicial H -homomorphism (with everything else in sight simplicial), then $h_*: \text{SMap}(X, P_1) \rightarrow \text{SMap}(X, P_2)$ defines a homomorphism $[h_*]: [X, P_1] \rightarrow [X, P_2]$.

Moreover, if μ, ν are locally effective (i.e., given $\sigma, \tau \in P$, we can evaluate $\mu(\sigma, \tau)$ and $\nu(\sigma)$) and X has finitely many nondegenerate simplices, then μ_*, ν_* are locally effective as well. Indeed, to specify a simplicial map $X \rightarrow P$, it suffices to give its values on the nondegenerate simplices, since on the degenerate simplices, it is determined uniquely; thus, such a simplicial map is a finite object, which can be handled algorithmically. (Moreover, in our algorithm, P is going to be a stage of a simplicial Postnikov tower, and simplicial maps $X \rightarrow P$ will have a compact representation by suitable vectors of cochains.)

Thus, under the above conditions, we have the Abelian group $[X, P_i]$ semi-effectively represented, where the set of representatives is $\text{SMap}(X, P)$. Similarly, if $h: P_1 \rightarrow P_2$ is locally effective and X has finitely many nondegenerate simplices, then $h_*: \text{SMap}(X, P_1) \rightarrow \text{SMap}(X, P_2)$ is locally effective, too.

A canonical H -group structure from connectivity. In our algorithm, the existence of a suitable H -group structure on P_i follows from the fact that P_i has nonzero homotopy groups only in the range from d to i , $i \leq 2d - 2$.

Lemma 4.2. *Let $d \geq 2$ and let P be a $(d - 1)$ -reduced CW complex with distinguished vertex (basepoint) o , and with nonzero $\pi_i(P)$ possibly occurring only for $i = d, d + 1, \dots, 2d - 2$. Then there are μ and ν such that (P, o, μ, ν) is an Abelian H -group, and moreover, o is a strictly neutral element, in the sense that $\mu(o, p) = \mu(p, o) = p$ (equalities, not only homotopy).*

Moreover, if μ' is any continuous binary operation on P with o as a strictly neutral element, then $\mu' \sim \mu$ by a homotopy stationary on the subspace $P \vee P := (P \times \{o\}) \cup (\{o\} \times P)$ (and, in particular, every such μ' automatically satisfies (HA), (HC), and (HI) with a suitable ν').

This lemma is essentially well-known, and the necessary arguments appear, e.g., in Whitehead [39]. We nonetheless sketch a proof, because we are not aware of a specific reference for the lemma as stated, and also because it sheds some light on how the assumption of $(d - 1)$ -connectedness of Y in Theorem 1.1 is used.

The proof is based on the repeated application of the following basic fact (which is a baby version of obstruction theory and can be proved by induction of the dimension of the cells on which the maps or homotopies have to be extended).

Fact 4.3. Suppose that X and Y are CW complexes, $A \subseteq X$ is a subcomplex, and assume that there is some integer k such that all cells in $X \setminus A$ have dimension at least k and that $\pi_i(Y) = 0$ for all $i \geq k - 1$. Then the following hold:

- (i) If $f: A \rightarrow Y$ is a continuous map, then there exists an extension $f': X \rightarrow Y$ of f (i.e., $f'|_A = f$).
- (ii) If $f \sim g: A \rightarrow Y$ are homotopic maps, and if $f', g': X \rightarrow Y$ are arbitrary extensions of f and of g , respectively, then $f' \sim g'$ (by a homotopy extending the given one on A).

Proof of Lemma 4.2. This proof is the *only* place where it is important that we work with CW-complexes, as opposed to simplicial sets. This is because the *product* of CW-complexes is defined differently from the product of simplicial sets. In the product of CW-complexes, an i -cell times a j -cell yields an $(i + j)$ -cell (and nothing else), while in products of simplicial sets, simplices of problematic intermediate dimensions appear.

Let $\varphi: P \vee P \rightarrow P$ be the *folding map* given by $\varphi(o, p) := p$, $\varphi(p, o) := p$, $p \in P$. Thus, the strict neutrality of o just means that μ extends φ , and we can employ Fact 4.3.

Namely, all cells in $(P \times P) \setminus (P \vee P)$ have dimension at least $2d$, and $\pi_i(P) = 0$ for $i \geq 2d - 1$. Thus, φ can be extended to some $\mu: P \times P \rightarrow P$, uniquely up to homotopy stationary on $P \vee P$.

From the homotopy uniqueness we get the homotopy commutativity (HC) immediately (for free). Indeed, if we define $\mu'(p, q) := \mu(q, p)$, then the homotopy uniqueness applies and yields $\mu' \sim \mu$. The homotopy associativity (HA) is also simple. Let $\psi_1, \psi_2: P^3 \rightarrow P$ be given by $\psi_1(p, q, r) := \mu(\mu(p, q), r)$ and $\psi_2(p, q, r) := \mu(p, \mu(q, r))$. Then $\psi_1 = \psi_2$ on the subspace $P \vee P \vee P := (P \times \{o\} \times \{o\}) \cup (\{o\} \times P \times \{o\}) \cup (\{o\} \times \{o\} \times P)$. Since all cells in $(P \times P \times P) \setminus (P \vee P \vee P)$ are of dimension at least $2d$, Fact 4.3 gives $\psi_1 \sim \psi_2$.

The existence of a homotopy inverse is not that simple, and actually, we won't need it (since we will construct an inverse explicitly). For a proof, we thus refer to the literature: every 0-connected CW-complex with an operation satisfying (HA) and (HN) also satisfies (HI); see, e.g., [39, Theorem X.2.2, p. 461]. \square

4.2 A locally effective H -group structure on the Postnikov stages

Now we are in the setting of Theorem 1.1; in particular, Y is a $(d - 1)$ -connected simplicial set. Let P_i , $i \geq 0$, denote the i th stage of a locally effective simplicial Postnikov system for Y , as in Section 3 (we will consider only the first $2d - 2$ stages). Since Y is $(d - 1)$ -connected, P_0 through P_{d-1} are trivial (one-point), and each P_i is $(d - 1)$ -reduced (the interval $[d, 2d - 2]$ is sometimes called the *stable range* and we will occasionally refer to the corresponding P_i as the *stable stages* of the Postnikov system.)

By Lemma 4.2, we know that the stable stages possess a (canonical) H -group structure. But we need to define the underlying operations on P_i concretely as simplicial maps and, mainly, make them effective. Since P_i is typically an infinite object, we will have just *local effectivity*, i.e., the operations can be evaluated algorithmically on any given pair of simplices.

From now on, we will denote the “addition” operation on P_i by \boxplus_i , and use the infix notation $\sigma \boxplus_i \tau$. Similarly we write $\boxminus_i \sigma$ for the “inverse” of σ . For a more convenient notation, we also introduce a *binary* version of \boxminus_i by setting $\sigma \boxminus_i \tau := \sigma \boxplus_i (\boxminus_i \tau)$.

Preliminary considerations. We recall that an m -simplex of P_i is written as $(\sigma^0, \sigma^1, \dots, \sigma^i)$, with $\sigma^i \in C^i(\Delta^m; \pi_i(Y))$. Thus, its components are cochains. One potential source of confu-

sion is that we already *have* a natural addition of such cochains defined; they can simply be added componentwise, as effectively as one might ever wish.

However, this *cannot* be used as the desired addition \boxplus_i . The reason is that the Postnikov classes k_{i-1} are generally *nonlinear*, and thus k_{i-1} is typically not a homomorphism with respect to cochain addition. In particular, we recall that P_i was defined as the subset of $P_{i-1} \times E_i$ “cut out” by k_{i-1} , i.e., via $k_{i-1}(\sigma) = \delta\sigma^i$, where $\sigma = (\sigma^0, \dots, \sigma^{i-1})$. Therefore, P_i is usually not even *closed* under the cochain addition.

Our approach to define a suitable operation \boxplus_i is inductive. Suppose that we have already defined \boxplus_{i-1} on P_{i-1} . Then we will first define \boxplus_i on special elements of P_i of the form $(\sigma, 0)$, by just adding the σ ’s according to \boxplus_{i-1} and leaving 0 in the last component.

Another important special case of \boxplus_i is on elements of the form $(\sigma, \sigma^i) \boxplus_i (\mathbf{0}, \tau^i)$. In this case, in spite of the general warning above against the cochain addition, the last components *are* added as cochains: $(\sigma, \sigma^i) \boxplus_i (\mathbf{0}, \tau^i) = (\sigma, \sigma^i + \tau^i)$. The main result of this section constructs a locally effective \boxplus_i that extends the two special cases just discussed.

Let us remark that by definition, \boxplus_i and \boxminus_i , as simplicial maps, operate on simplices of every dimension m . However, in the algorithm, we will be using them only up to $m \leq 2d - 2$, and so in the sequel we always implicitly assume that the considered simplices satisfy this dimensional restriction.

The main result on \boxplus_i, \boxminus_i . The following proposition summarizes everything about \boxplus_i, \boxminus_i we will need.

Proposition 4.4. *Let Y be a $(d-1)$ -connected simplicial set, $d \geq 2$, and let $P_d, P_{d+1}, \dots, P_{2d-2}$ be the stable stages of a locally effective Postnikov system with $2d-2$ stages for Y . Then each P_i has an Abelian H -group structure, given by locally effective simplicial maps $\boxplus_i: P_i \times P_i \rightarrow P_i$ and $\boxminus_i: P_i \rightarrow P_i$ with the following additional properties:*

- (a) $(\sigma, \sigma^i) \boxplus_i (\mathbf{0}, \tau^i) = (\sigma, \sigma^i + \tau^i)$ for all $(\sigma, \sigma^i) \in P_i$ and $\tau^i \in L_i$ (we recall that $L_i = K(\pi_i, i)$.)
- (b) $\boxminus_i(\mathbf{0}, \sigma^i) = (\mathbf{0}, -\sigma^i)$ for all $\sigma^i \in L_i$.
- (c) The projection $p_i: P_i \rightarrow P_{i-1}$ is a strict homomorphism, i.e., $p_i(\sigma \boxplus_i \tau) = p_i(\sigma) \boxplus_{i-1} p_i(\tau)$ and $p_i(\boxminus_i \sigma) = \boxminus_{i-1} p_i(\sigma)$ for all $\sigma, \tau \in P_i$.
- (d) If, moreover, $i < 2d-2$, then the Postnikov class $k_i: P_i \rightarrow K_{i+2}$ is an H -homomorphism (with respect to \boxplus_i on P_i and the simplicial group operation $+$, addition of cocycles, on K_{i+2}).

As was announced above, the proof of this proposition proceeds by induction on i . The heart is an explicit and effective version of (d), which we state and prove as a separate lemma.

Lemma 4.5. *Let P_i be a $(d-1)$ -connected simplicial set, and let $\mathbf{0}, \boxplus_i, \boxminus_i$ be an Abelian H -group structure on P_i , with \boxplus_i, \boxminus_i locally effective. Let $k_i: P_i \rightarrow K_{i+2}$ be a simplicial map, where $i < 2d-2$. Then there is a locally effective simplicial map $A_i: P_i \rightarrow E_{i+1}$ such that, for all simplices σ, τ of equal dimension, $A_i(\sigma, \mathbf{0}) = A_i(\mathbf{0}, \tau) = 0$, and*

$$k_i(\sigma \boxplus_i \tau) = k_i(\sigma) + k_i(\tau) + \delta A_i(\sigma, \tau).$$

We recall that $\delta: E_{i+1} \rightarrow K_{i+2}$ is the simplicial map induced by the coboundary operator, and that a simplicial map $f: P_i \rightarrow K_{i+2}$ is nullhomotopic iff it is of the form $\delta \circ F$ for some $F: P_i \rightarrow E_{i+1}$ (see Lemma 3.3). Therefore, the map A_i is an “effective witness” for the nullhomotopy of the map $(\sigma, \tau) \mapsto k_i(\sigma \boxplus_i \tau) - k_i(\sigma) - k_i(\tau)$, and so it shows that k_i is an H -homomorphism.

We postpone the proof of the lemma, and prove the proposition first.

Proof of Proposition 4.4. As was announced above, we proceed by induction on i . As an inductive hypothesis, we assume that, for some $i < 2d - 2$, locally effective simplicial maps \boxplus_i, \boxminus_i providing an H -group structure on P_i have been defined satisfying (a)–(c) in the proposition.

This inductive hypothesis is satisfied in the base case $i = d$: in this case we have $P_d = L_d$, and \boxplus_d and \boxminus_d are the addition and additive inverse of cocycles (under which L_d is even a simplicial Abelian group). Then (a),(b) obviously hold and (c) is void.

In order to make the inductive step from i to $i + 1$, we first apply Lemma 4.5 for P_i, \boxplus_i , and k_i , which yields a locally effective simplicial map $A_i: P_i \times P_i \rightarrow E_{i+1}$ with $A_i(\sigma, \mathbf{0}) = A_i(\mathbf{0}, \tau) = 0$ and $k_i(\sigma \boxplus_i \tau) = k_i(\sigma) + k_i(\tau) + \delta A_i(\sigma, \tau)$, for all σ, τ . As was remarked after the lemma, this implies that k_i is an H -homomorphism with respect to \boxplus_i .

Next, using A_i , we define the operations $\boxplus_{i+1}, \boxminus_{i+1}$ on P_{i+1} . We set

$$(\sigma, \sigma^{i+1}) \boxplus_{i+1} (\tau, \tau^{i+1}) := (\sigma \boxplus_i \tau, \omega^{i+1}), \quad \text{where } \omega^{i+1} := \sigma^{i+1} + \tau^{i+1} + A_i(\sigma, \tau). \quad (4)$$

Why is \boxplus_{i+1} simplicial? Since \boxplus_i is simplicial, it suffices to consider the last component, and this is a composition of simplicial maps, namely, of projections, A_i , and the operation $+$ in the simplicial group E_{i+1} . Clearly, \boxplus_{i+1} is also locally effective.

We also need to check that P_{i+1} is closed under this \boxplus_{i+1} . We recall that, for $\sigma \in P_i$, the condition for $(\sigma, \sigma^{i+1}) \in P_{i+1}$ is $k_i(\sigma) = \delta \sigma^{i+1}$. Using this condition for $(\sigma, \sigma^{i+1}), (\tau, \tau^{i+1}) \in P_{i+1}$, together with $\sigma \boxplus_i \tau \in P_i$ (inductive assumption), and the property of k_i above, we calculate $k_i(\sigma \boxplus_i \tau) = k_i(\sigma) + k_i(\tau) + \delta A_i(\sigma, \tau) = \delta \sigma^{i+1} + \delta \tau^{i+1} + \delta A_i(\sigma, \tau) = \delta \omega^{i+1}$, and thus $(\sigma, \sigma^{i+1}) \boxplus_{i+1} (\tau, \tau^{i+1}) \in P_{i+1}$ as needed.

Part (a) of the proposition for \boxplus_{i+1} follows from (4) and the property $A_i(\mathbf{0}, \tau) = 0 = A_i(\sigma, \mathbf{0})$. In particular, $(\mathbf{0}, 0)$ is a strictly neutral element for \boxplus_{i+1} .

Moreover, as a continuous map, \boxplus_{i+1} fulfills the assumptions on μ' in Lemma 4.2, and thus it satisfies the axioms of an Abelian H -group operation.

Next, we define the inverse operation \boxminus_{i+1} by

$$\boxminus_{i+1}(\sigma, \sigma^{i+1}) := (\boxminus_i \sigma, -\sigma^{i+1} - A_i(\sigma, \boxminus_i \sigma)).$$

It is simplicial for the same reason as that for \boxplus_{i+1} , and by a computation similar to the one for \boxplus_{i+1} above, we verify that P_{i+1} is closed under \boxminus_{i+1} .

To verify that this \boxminus_{i+1} indeed defines a homotopy inverse to \boxplus_{i+1} , we check that it actually is a *strict* inverse. Inductively, we assume $\sigma \boxminus_i \sigma = \mathbf{0}$ for all $\sigma \in P_i$, and from the formulas defining \boxplus_{i+1} and \boxminus_{i+1} , we check that $(\sigma, \sigma^{i+1}) \boxminus_{i+1} (\sigma, \sigma^{i+1}) = (\mathbf{0}, 0)$. Another simple calculation yields (b) for \boxminus_{i+1} .

Part (c) for \boxplus_{i+1} and \boxminus_{i+1} follows from the definitions and from $A_i(\mathbf{0}, \mathbf{0}) = 0$. This finishes the induction step and proves the proposition. \square

Proof of Lemma 4.5. Here we will use (“locally”) some terminology concerning chain complexes (e.g., chain homotopy, homomorphism of chain complexes), for which we refer to the literature (standard textbooks, say [13]).

First we define the *nonadditivity map* $a_i: P_i \times P_i \rightarrow K_{i+2}$ by

$$a_i(\sigma, \tau) := k_i(\sigma \boxplus_i \tau) - k_i(\sigma) - k_i(\tau).$$

(Thus, the map a_i measures the failure of k_i to be strictly additive with respect to \boxplus_i .) We want to show that $a_i = \delta A_i$ for a locally effective A_i .

Let us remark that the *existence* of A_i can be proved by an argument similar to the one in Lemma 4.2. That argument works for CW-complexes, and as was remarked in the proof of that lemma, it is essential that the product of an i -cell and a j -cell is an $(i+j)$ -cell and *nothing else*. For simplicial sets the product is defined differently, and if we consider $P_i \times P_i$ as a simplicial set, we do get simplices of “unpleasant” intermediate dimensions there.

We will get around this using the *Eilenberg–Zilber reduction* (which is also one of the basic tools in effective homology—but we won’t need effective homology directly); here, we follow the exposition in [12] (see also [27, Sections 7.8 and 8.2]). Loosely speaking, it will allow us to convert the setting of the simplicial set $P_i \times P_i$ to a setting (a tensor product of chain complexes) where only terms of the “right” dimensions appear.

We note that A_i is defined on an infinite object, so we cannot compute it globally—we need a local algorithm for evaluating it, yet its answers have to be globally consistent over the whole computation.

First we present the Eilenberg–Zilber reduction for an arbitrary simplicial set P with basepoint (and single vertex) o . The reduction consists of three locally effective maps²² AW, EML and SHI that fit into the following diagram:

$$\begin{array}{ccc} C_*(P) \otimes C_*(P) & \xrightleftharpoons[\text{AW}]{\text{EML}} & C_*(P \times P) \end{array} \quad \text{SHI}$$

Here $C_*(\cdot)$ denotes the (normalized) chain complex of a simplicial set, with integer coefficients (so we omit the coefficient group in the notation). For brevity, chains of all dimensions are collected into a single structure (whence the star subscript), and \otimes is the tensor product. Thus, $(C_*(P) \otimes C_*(P))_n = \bigoplus_{i+j=n} C_i(P) \otimes C_j(P)$. The operators AW and EML are homomorphisms of chain complexes, while SHI is a *chain homotopy* operator raising the degree by $+1$. Thus, for each n , we have $\text{AW}_n: C_n(P \times P) \rightarrow (C_*(P) \otimes C_*(P))_n$, $\text{EML}_n: (C_*(P) \otimes C_*(P))_n \rightarrow C_n(P \times P)$, and $\text{SHI}_n: C_n(P \times P) \rightarrow C_{n+1}(P \times P)$.

We refer to [12, pp. 1212–1213] for explicit formulas for AW and EML in terms of the face and degeneracy operators. We give only the formula for SHI, since A_i will be defined using SHI_{i+1} , and we summarize the properties of AW, EML, SHI relevant for our purposes.

The operator SHI_n operates on n -chains on $P \times P$. The formula given below specifies its values on the “basic” chains of the form (σ^n, τ^n) ; here σ^n, τ^n are n -simplices of P , but (σ^n, τ^n) is interpreted as the chain with coefficient 1 on (σ^n, τ^n) and 0 elsewhere. The definition then extends to arbitrary chains by linearity.

²²The acronyms stand for the mathematicians *Alexander* and *Whitney*, *Eilenberg* and *Mac Lane*, and *Shih*, respectively.

Let p and q be non-negative integers. A (p, q) -shuffle (α, β) is a partition

$$\{\alpha_1 < \cdots < \alpha_p\} \cup \{\beta_1 < \cdots < \beta_q\}$$

of the set $\{0, 1, \dots, p+q-1\}$. Put

$$\text{sig}(\alpha, \beta) = \sum_{i=1}^p (\alpha_i - i + 1).$$

Let $\gamma = \{\gamma_i, \dots, \gamma_r\}$ be a set of integers. Then s_γ denotes the compositions of the degeneracy operators $s_{\gamma_1} \dots s_{\gamma_r}$ (the s_m are the degeneracy operators of P , and ∂_m are its face operators). The operator SHI is defined by

$$\text{SHI}(\sigma^0, \tau^0) = 0,$$

$$\text{SHI}(\sigma^m, \tau^m) = \sum_{T(m)} (-1)^{\epsilon(\alpha, \beta)} (s_{\bar{\beta} + \bar{m}} \partial_{m-q+1} \cdots \partial_m \sigma^m, s_{\alpha + \bar{m}} \partial_{\bar{m}} \cdots \partial_{m-q-1} \tau^m),$$

where $T(m)$ is the set of all $(p+1, q)$ -shuffles such that $0 \leq p+q \leq m-1$,

$$\begin{aligned} \bar{m} &= m - p - q, \quad \epsilon(\alpha, \beta) = \bar{m} - 1 + \text{sig}(\alpha, \beta), \\ \alpha + \bar{m} &= \{\alpha_1 + \bar{m}, \dots, \alpha_{p+1} + \bar{m}\}, \quad \bar{\beta} + \bar{m} = \{\bar{m} - 1, \beta_1 + \bar{m}, \dots, \beta_q + \bar{m}\}. \end{aligned}$$

The above formula shows that SHI_n is locally effective, in the sense that, if a chain $c_n \in C_n(P \times P)$ is given in a locally effective way (by an algorithm that can evaluate the coefficient for each given n -simplex of $P \times P$), then a similar algorithm is available for the $(n+1)$ -chain $\text{SHI}_n(c_n)$ as well.

The first fact we will need is that for every n , the maps satisfy the following identity (where ∂ denotes the boundary operator in $C_*(P \times P)$):

$$\text{id}_{C_n(P \times P)} - \text{EML}_n \circ \text{AW}_n = \text{SHI}_{n-1} \circ \partial + \partial \circ \text{SHI}_n. \quad (5)$$

This identity says that SHI_n is a chain homotopy between $\text{EML}_n \circ \text{AW}_n$ and the identity on $C_n(P \times P)$.

The second fact, which follows directly from the formulas in [12], is that the operators EML and SHI behave well with respect to the basepoint o and its degeneracies, in the following sense: For every n and for every (nondegenerate) n -dimensional simplex τ^n of P (regarded as a chain),

$$\text{EML}_n(o \otimes \tau^n) = \pm(o^n, \tau^n), \quad \text{EML}_n(\tau^n \otimes o) = \pm(\tau^n, o^n), \quad (6)$$

where o^n is the (unique) n -dimensional degenerate simplex obtained from o . The images in (6) lie in the subgroup $C_n(P \vee P) \subseteq C_n(P \times P)$. Moreover, the operator SHI_n maps $C_n(P \vee P)$ into $C_{n+1}(P \vee P)$, i.e., the chains $\text{SHI}(o^n, \tau^n)$ and $\text{SHI}(\tau^n, o^n)$ are linear combinations of simplices of the form (o^{n+1}, σ^{n+1}) and (σ^{n+1}, o^{n+1}) , respectively, where σ^{n+1} ranges over certain $(n+1)$ -dimensional simplices of P .

We now apply this to $P = P_i$ (with basepoint $\mathbf{0}$). We consider the nonadditivity map a_i as an $(i+2)$ -cocycle on $P_i \times P_i$, which can be regarded as a homomorphism $a_i: C_{i+2}(P_i \times P_i) \rightarrow \pi_{i+1}$. If we compose this homomorphism a_i on the left with both sides of the identity (5), for $n = i+2$, we get

$$a_i \circ \text{id}_{C_{i+2}(P \times P)} - a_i \circ \text{EML}_{i+2} \circ \text{AW}_{i+2} = a_i \circ \text{SHI}_{i+1} \circ \partial + a_i \circ \partial \circ \text{SHI}_{i+2}. \quad (7)$$

Now $a_i \circ \partial = 0$ since a_i is a cocycle. Moreover, every basis element of $C_*(P_i) \otimes C_*(P_i)$ in degree $i+2 < 2d$ is of the form $\mathbf{0} \otimes \tau^{i+2}$ or $\tau^{i+2} \otimes \mathbf{0}$ (since P_i has no nondegenerate simplices in dimensions $1, \dots, d-1$). Such elements are taken by EML into $C_{i+1}(P \vee P)$, on which a_i vanishes because $\mathbf{0}$ is a strictly neutral element for \boxplus_i . Thus, $a_i \circ \text{EML}_{i+2} = 0$ for $i+2 < 2d$.

Therefore, (7) simplifies to $a_i = a_i \circ \text{SHI}_{i+1} \circ \partial$. Thus, if we set $A_i := a_i \circ \text{SHI}_{i+1}$, then $a_i = \delta A_i$, as desired (since applying δ to a cochain α corresponds to the composition $\alpha \circ \partial$ on the level of homomorphisms from chains into π_{i+1}). Finally, the property $A_i(\mathbf{0}, \cdot) = A_i(\cdot, \mathbf{0}) = 0$ follows because the corresponding property holds for a_i and SHI_{i+1} maps $C_{i+1}(P_i \vee P_i)$ to $C_{i+2}(P_i \vee P_i)$. \square

4.3 A semi-effective representation of $[X, P_i]$

Now let X be a finite simplicial complex or, more generally, a simplicial set with finitely many nondegenerate simplices (as we will see, the greater flexibility offered by simplicial sets will be useful in our algorithm, even if we want to prove Theorem 1.1 only for simplicial complexes X).

Having the locally effective H -group structure on the stable Postnikov stages P_i , we obtain the desired locally effective Abelian group structure on $[X, P_i]$ immediately.

Indeed, according to the remarks following Fact 4.1, a simplicial map $s: P \rightarrow Q$ of arbitrary simplicial sets induces a map $s_*: \text{SMap}(X, P) \rightarrow \text{SMap}(X, Q)$ by composition, i.e., by $s_*(f)(\sigma) = (s \circ f)(\sigma)$ for each simplex $\sigma \in P$. If P and Q are Kan, we also get a well-defined map $[s_*]: [X, P] \rightarrow [X, Q]$. Moreover, if s is locally effective, then so is s_* (since X has only finitely many nondegenerate simplices). In particular, the group operations on $[X, P_i]$ are represented by locally effective maps $\boxplus_{i*}: \text{SMap}(X, P_i) \times \text{SMap}(X, P_i) \rightarrow \text{SMap}(X, P_i)$ and $\boxminus_{i*}: \text{SMap}(X, P_i) \rightarrow \text{SMap}(X, P_i)$.

The cochain representation. However, we can make the algorithm considerably more efficient if we use the special structure of P_i and work with cochain representatives of the simplicial maps in $\text{SMap}(X, P_i)$.

We recall from Section 3 that simplicial maps into $K(\pi, n)$ and $E(\pi, n)$ are canonically represented by cocycles and cochains, respectively. Simplicial maps $X \rightarrow P_i$ are, in particular, maps into the product $E_0 \times \dots \times E_i$, and so they can be represented by $(i+1)$ -tuples of cochains $\mathbf{c} = (c^0, \dots, c^i)$, with $c^j \in C^j := C^j(X; \pi_j)$.

The “simplicial” definition of $\boxplus_{i*}, \boxminus_{i*}$ can easily be translated to a “cochain” definition, using the correspondence explained after Lemma 3.2. For simplicity, we describe the result concretely for the unary operation \boxminus_{i*} ; the case of \boxplus_{i*} is entirely analogous, it just would require more notation.

Thus, to evaluate $(d^0, \dots, d^i) := \boxminus_{i*} \mathbf{c}$, we need to compute the value of d^j on each j -simplex ω of X , $j = 0, 1, \dots, i$. To this end, we first identify ω with the standard j -simplex Δ^j via the unique order-preserving map of vertices. Then the restriction of (c^0, \dots, c^i) to ω (i.e., a labeling of the faces of ω by the elements of the appropriate Abelian groups) can be regarded as a j -simplex σ of P_i . We compute $\tau := \boxminus_j \sigma$, again a j -simplex of P_i . The component τ^j of τ is a j -cochain on Δ^j , i.e., a single element of π_j , and this value, finally, is the desired value of $d^j(\omega)$. For \boxplus_{i*} everything works similarly.

We also get that $\mathbf{0} \in \text{SMap}(X, P_i)$, the simplicial map represented by the zero cochains, is a strictly neutral element under \boxplus_{i*} .

We have made $[X, P_i]$ into a *semi-effectively represented Abelian group* in the sense of Section 2. The representatives are the $(i+1)$ -tuples (c^0, \dots, c^i) of cochains as above. However,

our state of knowledge of $[X, P_i]$ is rather poor at this point; for example, we have as yet no equality test.

A substantial amount of work still lies ahead to make $[X, P_i]$ fully effective.

5 The main algorithm

In order to prove our main result, Theorem 1.1, on computing $[X, Y]$, we will prove the following statement by induction on i .

Theorem 5.1. *Let X be a simplicial set with finitely many nondegenerate simplices, and let Y be a $(d-1)$ -connected simplicial set, $d \geq 2$, for which a locally effective Postnikov system with $2d-2$ stages P_0, \dots, P_{2d-2} is available. Then, for every $i = d, d+1, \dots, 2d-2$, a fully effective representation of $[X, P_i]$ can be computed, with the cochain representations of simplicial maps $X \rightarrow P_i$ as representatives.*

Two comments on this theorem are in order. First, unlike in Theorem 1.1, there is no restriction on $\dim X$ (the assumption $\dim X \leq 2d-2$ in Theorem 1.1 is needed only for the isomorphism $[X, Y] \cong [X, P_{2d-2}]$). Second, as was already mentioned in Section 4.3, even if we want Theorem 1.1 only for a simplicial *complex* X , we need Theorem 5.1 with simplicial *sets* X , because of recursion.

First we will (easily) derive Theorem 1.1 from Theorem 5.1.

Proof of Theorem 1.1. Given a Y as in Theorem 1.1, we first obtain a fully effective Postnikov system for it with $2d-2$ stages using Theorem 3.5. Then we compute a fully effective representation of $[X, P_{2d-2}]$ by Theorem 5.1. Since Y is $(d-1)$ -connected and $\dim X \leq 2d-2$, there is a bijection between $[X, Y]$ and $[X, P_{2d-2}]$ by Proposition 3.4.

It remains to implement the homotopy testing. Given two simplicial maps $f, g: X \rightarrow Y$, we use the locally effective simplicial map $\varphi_{2d-2}: Y \rightarrow P_{2d-2}$ (which is a part of a locally effective simplicial Postnikov system), and we compute the cochain representations \mathbf{c}, \mathbf{d} of the corresponding simplicial maps $\varphi_{2d-2} \circ f, \varphi_{2d-2} \circ g: X \rightarrow P_{2d-2}$. Then we can check, using the fully effective representation of $[X, P_{2d-2}]$, whether $[\mathbf{c}] - [\mathbf{d}] = 0$ in $[X, P_{2d-2}]$. This yields the promised homotopy testing algorithm for $[X, Y]$ and concludes the proof of Theorem 1.1. \square

5.1 The inductive step: An exact sequence for $[X, P_i]$

Theorem 5.1 is proved by induction on i . The base case is $i = d$ (since P_0, \dots, P_{d-1} are trivial for a $(d-1)$ -connected Y), which presents no problem: we have $P_d = L_d = K(\pi_d, d)$, and so

$$[X, P_d] \cong H^d(X; \pi_d).$$

This group is fully effective, since it is the cohomology group of a simplicial set with finitely many nondegenerate simplices, with coefficients in a fully effective group. (Alternatively, we could start the algorithm at $i = 0$; then it would obtain $[X, P_d]$ at stage d as well.)

So now we consider $i > d$, and we assume that a fully effective representation of $[X, P_{i-1}]$ is available, where the representatives of the homotopy classes $[f] \in [X, P_{i-1}]$ are (cochain representations of) simplicial maps $f: X \rightarrow P_{i-1}$. We want to obtain a similar representation for $[X, P_i]$.

Let us first describe on an intuitive level what this task means and how we are going to approach it.

As we know, every map $g \in \text{SMap}(X, P_i)$ yields a map $f = p_{i*}(g) = p_i \circ g \in \text{SMap}(X, P_{i-1})$ by projection (forgetting the last coordinate in P_i). We first ask the question of *which* maps $f \in \text{SMap}(X, P_{i-1})$ are obtained as such projections; this is traditionally called the *lifting problem* (and g is called a *lift* of f). Here the answer follows easily from the properties of the Postnikov system: liftability of a map f depends only on its homotopy class $[f] \in [X, P_{i-1}]$, and the liftable maps in $[X, P_{i-1}]$ are obtained as the kernel of the homomorphism $[k_{(i-1)*}]$ induced by the Postnikov class. This is very similar to the one-step extension in the setting of obstruction theory, as was mentioned in the introduction. This step will be discussed in Section 5.2.

Next, a single map $f \in \text{SMap}(X, P_{i-1})$ may in general have many lifts g , and we need to describe their structure. This is reasonably straightforward to do on the level of *simplicial maps*. Namely, if $\mathbf{c} = (c^0, \dots, c^{i-1})$ is the cochain representation of f and g_0 is a fixed lift of f , with cochain representation (\mathbf{c}, c_0^i) , then it turns out that all possible lifts g of f are of the form (again in the cochain representation) $(\mathbf{c}, c_0^i + z^i)$, $z^i \in Z^i(X, \pi_i) \cong \text{SMap}(X, L_i)$. Thus, all of these lifts have a simple “coset structure”.

This allows us to compute a list of generators of $[X, P_i]$. We also need to find all *relations* of these generators, and for this, we need to be able to test whether two maps $g_1, g_2 \in \text{SMap}(X, P_i)$ are homotopic. This is somewhat more complicated, and we will develop a recursive algorithm for homotopy testing in Section 5.4.

Using the group structure, it suffices to test whether a given $g \in \text{SMap}(X, P_i)$ is nullhomotopic. An obvious necessary condition for this is nullhomotopy of the projection $f = p_i \circ g$, which we test recursively. Then, if $f \sim 0$, we \boxplus_{i*} -add a suitable nullhomotopic map to g , and this reduces the nullhomotopy test to the case where g has a cochain representation of the form $(\mathbf{0}, z^i)$, $z^i \in Z^i(X, \pi_i) \cong \text{SMap}(X, L_i)$.

Now $(\mathbf{0}, z^i)$ can be nullhomotopic, as a map $X \rightarrow P_i$, by an “obvious” nullhomotopy, namely, one “moving” only the last coordinate, or in other words, induced by a nullhomotopy in $\text{SMap}(X, L_i)$. But there may also be “less obvious” nullhomotopies, and it turns out that these correspond to maps $SX \rightarrow P_{i-1}$, where SX is the suspension of X defined in Section 3.1. Thus, in order to be able to test homotopy of maps $X \rightarrow P_i$, we also need to compute $[SX, P_{i-1}]$ recursively, using the inductive assumption, i.e., Theorem 5.1 for $i - 1$.

The exact sequence. We will organize the computation of $[X, P_i]$ using an *exact sequence*, a basic tool in algebraic topology and many other branches of mathematics. First we write the sequence down, including some as yet undefined symbols, and then we provide some explanations. It goes as follows:

$$[SX, P_{i-1}] \xrightarrow{[\mu_i]} [X, L_i] \xrightarrow{[\lambda_{i*}]} [X, P_i] \xrightarrow{[p_{i*}]} [X, P_{i-1}] \xrightarrow{[k_{(i-1)*}]} [X, K_{i+1}]. \quad (8)$$

This is a sequence of Abelian groups and homomorphisms of these groups, and exactness means that the image of each of the homomorphisms equals the kernel of the successive one.

We have already met most of the objects in this exact sequence, but for convenience, let us summarize them all.

- $[SX, P_{i-1}]$ is the group of homotopy classes of maps from the suspension into the one lower stage P_{i-1} ; inductively, we may assume it to be fully effective.
- $[\mu_i]$ is a homomorphism appearing here for the first time, which will be discussed later.
- $[X, L_i] \cong H^i(X; \pi_i)$ consists of the homotopy classes of maps into the Eilenberg–MacLane space $L_i = K(\pi_i, i)$, and it is fully effective.

- $[\lambda_{i*}]$ is the homomorphism induced by the mapping $\lambda_i: L_i \rightarrow P_i$, the “insertion to the last component”; i.e., $\lambda_i(\sigma^i) = (\mathbf{0}, \sigma^i)$. In terms of cochain representatives, λ_{i*} sends z^i to $(\mathbf{0}, z^i)$.
- $[X, P_i]$ is what we want to compute, $[p_{i*}]$ is the projection (on the level of homotopy), and $[X, P_{i-1}]$ has already been computed, as a fully effective Abelian group.
- $[k_{(i-1)*}]$ is the homomorphism induced by the composition with the Postnikov class $k_{i-1}: P_{i-1} \rightarrow K_{i+1} = K(\pi_i, i+1)$.
- $[X, K_{i+1}] \cong H^{i+1}(X, \pi_i)$ are again maps into an Eilenberg–MacLane space.

Let us remark that the exact sequence (8), with some $[\mu_i]$, can be obtained by standard topological considerations from the so-called *fibration sequence* for the fibration $L_i \rightarrow P_i \rightarrow P_{i-1}$; see, e.g., [19, Chap. 14].²³ However, in order to have all the homomorphisms locally effective and also to provide the locally effective “inverses” (as required in Lemma 2.4), we will need to analyze the sequence in some detail; then we will obtain a complete “pedestrian” proof of the exactness with only a small extra effort. Thus, the fibration sequence serves just as a background.

The algorithm for computing $[X, P_i]$ goes as follows.

1. Compute $[X, P_{i-1}]$ fully effective, recursively.
2. Compute $N_{i-1} := \ker [k_{(i-1)*}] \subseteq [X, P_{i-1}]$ (so N_{i-1} consists of all homotopy classes of liftable maps), fully effective, using Lemma 2.2 and Theorem 3.5.
3. Compute $[SX, P_{i-1}]$ fully effective, recursively.
4. Compute the factor group $M_i := \text{coker} [\mu_i] = [X, L_i] / \text{im} [\mu_i]$ using Lemma 2.3, fully effective and including the possibility of computing “witnesses for 0” as in the lemma.
5. The exact sequence (8) can now be transformed to the short exact sequence

$$0 \rightarrow M_i \xrightarrow{\ell_i} [X, P_i] \xrightarrow{[p_{i*}]} N_{i-1} \rightarrow 0$$

²³Let us consider topological spaces E and B with basepoints and a pointed map $p: E \rightarrow B$. If p has the so-called *homotopy lifting property* (which is the case for our p_i) it is called a *fibration* and the preimage F of the base point in B is called the *fibre* of p . The sequence of maps $F \xrightarrow{i} E \xrightarrow{p} B$ can be prolonged into the *fibration sequence*

$$\cdots \rightarrow \Omega F \xrightarrow{\Omega i} \Omega E \xrightarrow{\Omega p} \Omega B \xrightarrow{\mu} F \xrightarrow{i} E \xrightarrow{p} B$$

of pointed maps, where, for a pointed space Y , ΩY is the space of loops starting at the base point. For spaces X and Y with base points, let $\text{Map}(X, Y)_*$ denote the set of all continuous pointed maps, and let $[X, Y]_*$ be the set of (pointed) homotopy classes of these maps. Then the fibration sequence yields the sequence

$$\cdots \rightarrow \text{Map}(X, \Omega F)_* \rightarrow \text{Map}(X, \Omega E)_* \rightarrow \text{Map}(X, \Omega B)_* \rightarrow \text{Map}(X, F)_* \rightarrow \text{Map}(X, E)_* \rightarrow \text{Map}(X, B)_*.$$

As it turns out, on the level of homotopy classes we get even the long *exact* sequence

$$\cdots \rightarrow [X, \Omega F]_* \rightarrow [X, \Omega E]_* \rightarrow [X, \Omega B]_* \rightarrow [X, F]_* \rightarrow [X, E]_* \rightarrow [X, B]_*.$$

There is a natural bijection between $[\Sigma X, E]_*$ and $[X, \Omega E]_*$, where ΣX is the *reduced* suspension of X , and so we get the long exact sequence

$$\cdots \rightarrow [\Sigma X, F]_* \rightarrow [\Sigma X, E]_* \rightarrow [\Sigma X, B]_* \rightarrow [X, F]_* \rightarrow [X, E]_* \rightarrow [X, B]_*.$$

For CW-complexes, the difference between SX and ΣX doesn’t matter, and for the sequence $P_i \rightarrow P_{i-1} \rightarrow K_{i+1}$, which can be considered as a fibration, we arrive at (8).

(where ℓ_i is induced by exactly the same mapping λ_{i*} of representatives as $[\lambda_{i*}]$ in the original exact sequence (8)). Let $\mathcal{N}_{i-1} := \{f \in \text{SMap}(X, P_{i-1}) : [k_{(i-1)*}(f)] = 0\}$ be the set of representatives of elements in N_{i-1} . Implement a locally effective “section” $\xi_i : \mathcal{N}_{i-1} \rightarrow \text{SMap}(X, P_i)$ with $[p_{i*} \circ \xi_i] = \text{id}$ and a locally effective “inverse” $r_i : \text{im}[\lambda_{i*}] \rightarrow M_i$ with $\ell_i \circ r_i = \text{id}$, as in Lemma 2.4, and compute $[X, P_i]$ fully effective using that lemma.

We will now examine steps 2,4,5 in detail, and simultaneously establish the exactness of (8).

Convention. It will be notationally convenient to let maps such as p_{i*} , $k_{(i-1)*}$, λ_{i*} , which send simplicial maps to simplicial maps, operate directly on the cochain representations (and in such case, the result is also assumed to be a cochain representation). Thus, for example, we can write $p_{i*}(\mathbf{c}, c) = \mathbf{c}$, $\lambda_{i*}(z^i) = (\mathbf{0}, z^i)$, etc. We also write $[\mathbf{c}]$ for the homotopy class of the map represented by \mathbf{c} .

5.2 Computing the liftable maps

Here we will deal with the last part of the exact sequence (8), namely,

$$[X, P_i] \xrightarrow{[p_{i*}]} [X, P_{i-1}] \xrightarrow{[k_{(i-1)*}]} [X, K_{i+1}].$$

First we note that, since the projection map p_i is an H -homomorphism by Proposition 4.4(c), the (locally effective) map $p_{i*} : \text{SMap}(X, P_i) \rightarrow \text{SMap}(X, P_{i-1})$ indeed induces a well-defined group homomorphism $[X, P_i] \rightarrow [X, P_{i-1}]$ (Fact 4.1). Similarly, the H -homomorphism k_{i-1} (Proposition 4.4(d)) induces a group homomorphism $[k_{(i-1)*}] : [X, P_{i-1}] \rightarrow [X, K_{i+1}] \cong H^{i+1}(X; \pi_i)$.

Lemma 5.2 (Lifting lemma). *We have $\text{im}[p_{i*}] = \ker[k_{(i-1)*}]$. Moreover, if we set $\mathcal{N}_{i-1} := \{f \in \text{SMap}(X, P_{i-1}) : [k_{(i-1)*}(f)] = 0\}$, then there is a locally effective mapping $\xi_i : \mathcal{N}_{i-1} \rightarrow \text{SMap}(X, P_i)$ such that $p_{i*} \circ \xi_i$ is the identity map (on the level of simplicial maps).*

Proof. Let us consider a map $f \in \text{SMap}(X, P_{i-1})$ with cochain representation \mathbf{c} . Every cochain (\mathbf{c}, c^i) with $c^i \in C^i(X; \pi_i)$ represents a simplicial map $X \rightarrow P_{i-1} \times E_i$, and this map goes into P_i iff the condition

$$k_{(i-1)*}(\mathbf{c}) = \delta c^i \tag{9}$$

holds. Thus, f has a lift iff $k_{(i-1)*}(\mathbf{c})$ is a coboundary, or in other words, iff $[k_{(i-1)*}(\mathbf{c})] = 0$ in $[X, K_{i+1}]$. Hence $\text{im}[p_{i*}] = \ker[k_{(i-1)*}]$ indeed.

Moreover, if $k_{(i-1)*}(\mathbf{c})$ is a coboundary, we can compute some c^i satisfying (9) and set $\xi_i(f) := (\mathbf{c}, c^i)$. This involves some arbitrary choice, but if we fix some (arbitrary) rule for choosing c^i , we obtain a locally effective ξ_i as needed. The lemma is proved. \square

We have thus proved exactness of the sequence (8) at $[X, P_{i-1}]$. Step 2 of the algorithm can be implemented using Lemma 2.2. We have also prepared the section ξ_i for Step 5.

5.3 Factoring by maps from SX

We now focus on the initial part

$$[SX, P_{i-1}] \xrightarrow{[\mu_i]} [X, L_i] \xrightarrow{[\lambda_{i*}]} [X, P_i]$$

of the exact sequence (8), and explain how the suspension comes into the picture. We remark that $[\lambda_{i*}]$ is a well-defined homomorphism, for the same reason as $[p_{i*}]$ and $[k_{(i-1)*}]$; namely, λ_i is an H -homomorphism by Proposition 4.4(a).

The kernel of $[\lambda_{i*}]$ describes all homotopy classes of maps $X \rightarrow L_i$ that are nullhomotopic as maps $X \rightarrow P_i$. To understand how they arise as images of maps $SX \rightarrow P_{i-1}$, we first need to discuss a representation of nullhomotopies as maps from the cone.

Maps from the cone. A map $X \rightarrow Y$ between two topological spaces is nullhomotopic iff it can be extended to a map $CX \rightarrow Y$ on the cone over X ; this is more or less a reformulation of the definition of nullhomotopy. The same is true in the simplicial setting if the target is a Kan simplicial set, such as P_i .

We recall that the n -dimensional nondegenerate simplices of CX are of two kinds: the n -simplices of X and the cones over the $(n-1)$ -simplices of X . In the language of cochains, this means that, for any coefficient group π , we have

$$C^n(CX; \pi) \cong C^{n-1}(X; \pi) \oplus C^n(X; \pi),$$

and thus a cochain $b \in C^n(CX; \pi)$ can be written as (e, c) , with $e \in C^{n-1}(X; \pi)$ and $c \in C^n(X; \pi)$. We also write $c = b|_X$ for the restriction of b to X . The coboundary operator $C^n(CX; \pi) \rightarrow C^{n+1}(CX; \pi)$ then acts as follows:

$$\delta(e, c) = (-\delta e + c, \delta c).$$

Rephrasing Lemma 3.3 in the language of extensions to CX , we get the following:

Corollary 5.3. *A map $f \in \text{SMap}(X, L_i)$, represented by a cocycle $c^i \in Z^i(X; \pi_i)$, is nullhomotopic iff there is a cocycle $b \in Z^i(CX; \pi) \cong \text{SMap}(CX, L_i)$ such that $b|_X = c$.*

This describes the homotopies in $\text{SMap}(X, L_i)$, which induce the “obvious” homotopies in $\text{im } \lambda_{i*}$. Let us now consider an element in the image of λ_{i*} , i.e., a map $g: X \rightarrow P_i$ with a cochain representation $(\mathbf{0}, c^i)$. By the above, a nullhomotopy of g can be regarded as a simplicial map $G: CX \rightarrow P_i$ whose cochain representation (\mathbf{b}, b^i) satisfies $(\mathbf{b}|_X, b^i|_X) = (\mathbf{0}, c^i)$ (here $\mathbf{b}|_X = (b^0|_X, \dots, b^{i-1}|_X)$ is the componentwise restriction to X). Thus, the projection $F := p_{i*} \circ G \in \text{SMap}(CX, P_{i-1})$ is represented by \mathbf{b} with $\mathbf{b}|_X = \mathbf{0}$, and hence it maps all of the “base” X in CX to $\mathbf{0}$.

Recalling that SX is obtained from CX by identifying X to a single vertex, we can see that such F exactly correspond to simplicial maps $SX \rightarrow P_{i-1}$ (here we use that P_{i-1} has a single vertex $\mathbf{0}$). Thus, maps in $\text{SMap}(SX, P_{i-1})$ give rise to nullhomotopies of maps in $\text{im } \lambda_{i*}$.

After this introduction, we develop the definition of μ_i and prove the exactness of our sequence (8) at $[X, L_i]$.

The homomorphism μ_i . Since the nondegenerate $(i+1)$ -simplices of SX are in one-to-one correspondence with the nondegenerate i -simplices of X , we have the isomorphism of the cochain groups

$$D_i: C^{i+1}(SX; \pi_i) \rightarrow C^i(X; \pi_i).$$

Moreover, this is compatible with the coboundary operator (up to sign):

$$\delta D_i(c) = -D_i(\delta c).$$

Alternatively, if we identify the $(i+1)$ -cochains on SX with those $(i+1)$ -cochains $b = (e, c) \in C^{i+1}(CX; \pi_i)$ for which $b|_X = c = 0$, then the isomorphism is given by $D_i(e, 0) = e$. The coboundary formula $\delta(e, c) = (-\delta e + c, \delta c)$ for CX indeed gives $D_i(\delta(e, 0)) = D_i(-\delta e, 0) = -\delta e = -\delta D_i(e, 0)$.

Because of the compatibility with δ , D_i restricts to an isomorphism $Z^{i+1}(SX; \pi_i) \rightarrow Z^i(X; \pi_i)$ (which we also denote by D_i). This induces an isomorphism $[D_i]: H^{i+1}(SX; \pi_i) \rightarrow H^i(X; \pi_i)$.

Translating from cochains to simplicial maps, we can also regard D_i as an isomorphism $\text{SMap}(SX, K_{i+1}) \rightarrow \text{SMap}(X, L_i)$, (where, as we recall, $K_{i+1} = K(\pi_i, i+1)$ and $L_i = K(\pi_i, i)$), and $[D_i]$ as an isomorphism $[SX, K_{i+1}] \rightarrow [X, L_i]$.

Now we define $\mu_i: \text{SMap}(SX, P_{i-1}) \rightarrow \text{SMap}(X, L_i)$ by

$$\mu_i := D_i \circ k_{(i-1)*}.$$

That is, given $F \in \text{SMap}(SX, P_{i-1})$, we first compose it with k_{i-1} , which yields a map in $\text{SMap}(SX, K_{i+1})$ represented by a cocycle in $Z^{i+1}(SX; \pi_i)$. Applying D_i means re-interpreting this as a cocycle in $Z^i(X; \pi_i)$ representing a map in $\text{SMap}(X, L_i)$, which we declare to be $\mu_i(F)$. This, clearly, is locally effective, and $[\mu_i]$ is a well-defined homomorphism $[SX, P_{i-1}] \rightarrow [X, L_i]$ (since $[D_i]$ and $[k_{(i-1)*}]$ are well-defined homomorphisms).

The connection of this definition of μ_i to the previous considerations on nullhomotopies may not be obvious at this point, but the lemma below shows that μ_i works.

Lemma 5.4. *The sequence (8) is exact at $[X, L_i]$, i.e., $\text{im}[\mu_i] = \ker[\lambda_{i*}]$.*

Proof. First we want to prove the inclusion $\text{im}[\mu_i] \subseteq \ker[\lambda_{i*}]$. To this end, we consider $F \in \text{SMap}(SX, P_{i-1})$ arbitrary and want to show that $[\lambda_{i*}(\mu_i(F))] = 0$ in $[X, P_i]$.

As was discussed above, we can view F as a map $\overline{F}: CX \rightarrow P_{i-1}$ that is zero on X . Let \mathbf{b} be the cochain representation of \overline{F} ; thus, $\mathbf{b}|_X = \mathbf{0}$.

Let $z^i \in Z^i(X; \pi_i)$ be the cocycle representing $\mu_i(F)$. Then $(0, z^i) \in C^{i-1}(X; \pi_i) \oplus C^i(X; \pi_i)$ represents a map $CX \rightarrow E_i$, and $(\mathbf{b}, (0, z^i))$ represents a map $G: CX \rightarrow P_{i-1} \times E_i$.

We claim that G actually goes into P_i , i.e., is a lift of \overline{F} . For this, we just need to verify the lifting condition (9), which reads $k_{(i-1)*}(\mathbf{b}) = \delta(0, z^i)$.

By the coboundary formula for the cone, we have $\delta(0, z^i) = (z^i, 0)$, while $k_{(i-1)*}(\mathbf{b}) = (z^i, 0)$ by the definition of $\mu_i(F)$. So $G \in \text{SMap}(CX, P_i)$ is indeed a lift of \overline{F} . At the same time, $(\mathbf{b}, (0, z^i))|_X = (\mathbf{0}, z^i)$, and so G is a nullhomotopy for the map represented by $(\mathbf{0}, z^i)$, which is just $\lambda_{i*}(\mu_i(F))$.

To prove the reverse inclusion $\text{im}[\mu_i] \supseteq \ker[\lambda_{i*}]$, we proceed similarly. Suppose that $z^i \in Z^i(X; \pi_i)$ represents a map $f \in \text{SMap}(X, L_i)$ with $[\lambda_{i*}(f)] = 0$ in $[X, P_i]$. Then $\lambda_{i*}(f)$ has the cochain representation $(\mathbf{0}, z^i)$, and there is a nullhomotopy $G \in \text{SMap}(CX, P_i)$ for it, with a cochain representation $(\mathbf{b}, (a^{i-1}, z^i))$, where $\mathbf{b}|_X = \mathbf{0}$.

Since $\mathbf{b}|_X = \mathbf{0}$, \mathbf{b} represents a map $\overline{F} \in \text{SMap}(CX, P_{i-1})$ zero on X , which can also be regarded as $F \in \text{SMap}(SX, P_{i-1})$. Let \tilde{z}^i represent $\mu_i(F)$. Since G is a lift of \overline{F} , the lifting condition $k_{(i-1)*}(\mathbf{b}) = \delta(a^{i-1}, z^i)$ holds. We have $k_{(i-1)*}(\mathbf{b}) = (\tilde{z}^i, 0)$, again by the definition of μ_i , and $\delta(a^{i-1}, z^i) = (-\delta a^{i-1} + z^i, \delta z^i)$ by the coboundary formula for the cone. Hence $\tilde{z}^i - z^i = \delta a^{i-1}$, which means that $[z^i] = [\tilde{z}^i]$. Thus $[f] = [\mu_i(F)] \in \text{im}[\mu_i]$, and the lemma is proved. \square

Having $[\mu_i]$ defined as a locally effective homomorphism, we can employ Lemma 2.3 and implement Step 4 of the algorithm.

5.4 Computing nullhomotopies

The next step is to prove the exactness of the sequence (8) at $[X, P_i]$.

Lemma 5.5. *We have $\text{im } [\lambda_{i*}] = \ker [p_{i*}]$.*

Proof. The inclusion $\text{im } [\lambda_{i*}] \subseteq \ker [p_{i*}]$ holds even on the level of simplicial maps, i.e., $\text{im } \lambda_{i*} \subseteq \ker p_{i*}$. Indeed, $p_{i*}(\lambda_{i*}(z^i)) = p_{i*}(\mathbf{0}, z^i) = \mathbf{0}$.

For the reverse inclusion, consider $(\mathbf{c}, c^i) \in \text{SMap}(X, P_i)$ and suppose that $[p_{i*}(\mathbf{c}, c^i)] = [\mathbf{c}] = 0 \in [X, P_{i-1}]$. We need to find some $z^i \in Z^i(X; \pi_i)$ with $[(\mathbf{0}, z^i)] = [(\mathbf{c}, c^i)]$ in $[X, P_i]$.

A suitable z^i can be constructed by taking a nullhomotopy $CX \rightarrow P_{i-1}$ for \mathbf{c} and lifting it. Namely, let \mathbf{b} represent a nullhomotopy for \mathbf{c} , i.e., $\mathbf{b}|_X = \mathbf{c}$, and let (\mathbf{b}, b^i) be a lift of \mathbf{b} (it exists because CX is contractible and thus *every* map on it can be lifted). We set

$$z^i := c^i - (b^i|_X).$$

We need to verify that z^i is a cocycle. This follows from the lifting conditions $k_{(i-1)*}(\mathbf{c}) = \delta c^i$ and $k_{(i-1)*}(\mathbf{b}) = \delta b^i$, and from the fact that $k_{(i-1)*}(\mathbf{b})|_X = k_{(i-1)*}(\mathbf{b}|_X) = k_{(i-1)*}(\mathbf{c})$ (this is because applying $k_{(i-1)*}$ really means a composition of maps, and thus it commutes with restriction). Indeed, we have $\delta z^i = \delta c^i - \delta(b^i|_X) = k_{(i-1)*}(\mathbf{c}) - k_{(i-1)*}(\mathbf{c}) = 0$.

It remains to check that $[(\mathbf{c}, c^i)] = [(\mathbf{0}, z^i)]$. We calculate $[(\mathbf{c}, c^i)] - [(\mathbf{0}, z^i)] = [(\mathbf{c}, c^i) \boxminus_{i*} (\mathbf{0}, z^i)] = [(\mathbf{c}, c^i - z^i)] = [(\mathbf{c}, b^i|_X)] = [(\mathbf{b}|_X, b^i|_X)] = 0$, since (\mathbf{b}, b^i) is a nullhomotopy for $(\mathbf{b}|_X, b^i|_X)$. \square

Defining the inverse for λ_{i*} . Now we consider the cokernel $M_i = [X, L_i] / \text{im } [\mu_i]$ as in Step 4 of the algorithm, and the (injective) homomorphism $\ell_i: M_i \rightarrow [X, P_i]$ induced by $[\lambda_{i*}]$.

The last thing we need for applying Lemma 2.4 in Step 5 is a locally effective map $r_i: \text{im } \ell_i \rightarrow M_i$ with $\ell_i \circ r_i = \text{id}$.

Let \mathcal{R}_i be the set of representatives of the elements in $\text{im } \ell_i = \text{im } [\lambda_{i*}]$; by the above, we can write $\mathcal{R}_i = \{(\mathbf{c}, c^i) \in \text{SMap}(X, P_i) : [\mathbf{c}] = 0\}$.

For every $(\mathbf{c}, c^i) \in \mathcal{R}_i$ we set

$$\rho_i(\mathbf{c}, c^i) := z^i,$$

where z^i is as in the above proof of Lemma 5.5 (i.e., $z^i = c^i - (b^i|_X)$, where (\mathbf{b}, b^i) is a lifting of some nullhomotopy \mathbf{b} for \mathbf{c}). This definition involves choice of a particular \mathbf{b} and b^i , which we make arbitrarily for each (\mathbf{c}, c^i) .

Lemma 5.6. *The map ρ_i induces a map $r_i: \text{im } [\lambda_{i*}] \rightarrow [X, L_i]$ such that $\ell_i \circ r_i = \text{id}$.*

Proof. In the proof of Lemma 5.5 we have verified that $[\lambda_{i*}(\rho_i(\mathbf{c}, c^i))] = [(\mathbf{c}, c^i)]$, so $\lambda_{i*} \circ \rho_i$ acts as the identity on the level of homotopy classes. It follows that r_i is well-defined, since ℓ_i is injective and thus the condition $\ell_i \circ r_i = \text{id}$ determines r_i uniquely. \square

We note that, since we assume $[X, P_{i-1}]$ fully effective, we can algorithmically test whether $[\mathbf{c}] = 0$, i.e., whether \mathbf{c} represents a nullhomotopic map—the problem is in computing a concrete nullhomotopy \mathbf{b} for \mathbf{c} .

We describe a recursive algorithm for doing that. For more convenient notation, we will formulate it for computing nullhomotopies for maps in $\text{SMap}(X, P_i)$, but we note that, when evaluating ρ_i , we actually use this algorithm with $i - 1$ instead of i . Actually, some of the ideas in the algorithm are very similar to those in the proof of the exactness at $[X, P_i]$ (Lemma 5.5 above), so we could have started with a presentation of the algorithm instead of Lemma 5.5, but we hope that a more gradual development may be easier to follow.

The nullhomotopy algorithm. So now we formulate a recursive algorithm $\text{NullHom}(\mathbf{c}, c^i)$, which takes as input a cochain representation of a nullhomotopic map in $\text{SMap}(X, P_i)$ (i.e., such that $[(\mathbf{c}, c^i)] = 0$), and outputs a nullhomotopy (\mathbf{b}, b^i) for (\mathbf{c}, c^i) .

The required nullhomotopy (\mathbf{b}, b^i) will be \boxplus_{i*} -added together from several nullhomotopies; this decomposition is guided by the left part of our exact sequence (8). Namely, we recursively find a nullhomotopy for \mathbf{c} and lift it, which reduces the original problem to finding a nullhomotopy for a map in $\text{im } \lambda_{i*}$, of the form $(\mathbf{0}, z^i)$ (as in the proof of Lemma 5.5). Then, using the fact that ℓ_i is an isomorphism, we find nullhomotopies witnessing that $[z^i] = 0$ in M_i (here we need the assumption that the representation of M_i allows for computing “witnesses of zero” as in Lemma 2.3).

For this to work, we need the fact that if \mathbf{b}_1 is a nullhomotopy for \mathbf{c}_1 and \mathbf{b}_2 is a nullhomotopy for \mathbf{c}_2 , then $\mathbf{b}_1 \boxplus_{i*} \mathbf{b}_2$ is a nullhomotopy for $\mathbf{c}_1 \boxplus_{i*} \mathbf{c}_2$. This is because \boxplus_{i*} operates on mappings by composition, and thus it commutes with restrictions—we have already used the same observation for k_{i*} .

The base case of the algorithm is $i = d$. Here, as we recall, $P_d = L_d = K(\pi_d, d)$, and a nullhomotopic c^d means that $c^d \in Z^d(X; \pi_d)$ is a coboundary. We thus compute $e \in Z^{d-1}(X; \pi_d)$ with $c^d = \delta e$, and the desired nullhomotopy is $(e, \delta e)$ (indeed, $(e, \delta e)$ specifies a valid map $CX \rightarrow L_d$ since, by the coboundary formula for the cone, it is a cocycle).

Now we can state the algorithm formally.

Algorithm $\text{NullHom}(\mathbf{c}, c^i)$.

- A. (Base case) If $i = d$, return $(\mathbf{b}, b^d) = (\mathbf{0}, (e, \delta e))$ as above and stop.
- B. (Recursion) Now $i > d$. Set $\mathbf{b}_0 := \text{NullHom}(\mathbf{c})$, and let (\mathbf{b}_0, b_0^i) be an arbitrary lift of \mathbf{b}_0 .
- C. (Nullhomotopy coming from SX) Set $z^i := c^i - (b_0^i|_X)$, and use the representation of M_i to find a “witness for $[z^i] = 0$ in M_i ”. That is, compute $F \in [SX, P_{i-1}]$ such that $[z^i] = [\tilde{z}^i]$ in $[X, L_i]$, where \tilde{z}^i is the cocycle representing $\mu_i(F)$. Let \mathbf{a} be the cochain representation of the map $\overline{F} \in \text{SMap}(CX, P_{i-1})$ corresponding to F .
- D. (Nullhomotopy in $[X, L_i]$) Compute $e \in Z^{i-1}(X; \pi_i)$ with $\tilde{z}^i - z^i = \delta e$. (Then, as in the base case $i = d$ above, $(e, \delta e)$ is a nullhomotopy for $\tilde{z}^i - z^i$, and thus $(\mathbf{0}, (e, \delta e))$ is a nullhomotopy for $(\mathbf{0}, \tilde{z}^i - z^i)$.)
- E. Return

$$(\mathbf{b}, b^i) := (\mathbf{b}_0, b_0^i) \boxplus_{i*} \left((\mathbf{a}, (0, \tilde{z}^i)) \boxplus_{i*} (\mathbf{0}, (e, \delta e)) \right).$$

Proof of correctness. First we need to check that z^i in Step C indeed represents 0 in M_i . This is because, as in the proof of Lemma 5.5, $[(\mathbf{0}, z^i)] = [\lambda_{i*}(z^i)] = 0$, and since ℓ_i is injective, we have $[z^i] = 0$ in M_i as claimed. So the algorithm succeeds in computing some (\mathbf{b}, b^i) , and we just need to check that it is a nullhomotopy for (\mathbf{c}, c^i) .

All three terms in the formula in Step E are valid representatives of maps $CX \rightarrow P_i$ (for (\mathbf{b}_0, b_0^i) this follows from the inductive hypothesis, for $(\mathbf{a}, (0, \tilde{z}^i))$ we have checked this in the first part of the proof of Lemma 5.4, and for $(\mathbf{0}, (e, \delta e))$ we have already discussed this). So (\mathbf{b}, b^i) also represents such a map, and all we need to do is to check that $(\mathbf{b}|_X, b^i|_X) = (\mathbf{c}, c^i)$:

$$\begin{aligned} (\mathbf{b}|_X, b^i|_X) &= (\mathbf{b}_0|_X, b_0^i|_X) \boxplus_{i*} \left((\mathbf{a}|_X, \tilde{z}^i) \boxplus_{i*} (\mathbf{0}, \delta e) \right) \\ &= (\mathbf{c}, b_0^i|_X) \boxplus_{i*} \left((\mathbf{0}, \tilde{z}^i) \boxplus_{i*} (\mathbf{0}, z^i - \tilde{z}^i) \right) \\ &= (\mathbf{c}, b_0^i|_X + \tilde{z}^i + z^i - \tilde{z}^i) = (\mathbf{c}, b_0^i|_X + (c^i - (b_0^i|_X))) = (\mathbf{c}, c^i). \end{aligned}$$

Thus, the algorithm correctly computes the desired nullhomotopy. \square

As we have already explained, the algorithm makes ρ_i locally effective, and so Step 5 of the main algorithm can be implemented. This completes the proof of Theorem 5.1.

Remark. In order to compute $[X, P_i]$, our algorithm recursively computes all $[SX, P_j]$, $d \leq j \leq i-1$. If we take the algorithm literally, for computing $[SX, P_{i-1}]$ we should recursively compute $[SSX, P_{i-2}]$ etc., forming essentially a complete binary tree of recursive calls.

However, for our use of $[SX, P_j]$, we only need a set of *generators* for it, but we don't need the *relations*; see the remark following Lemma 2.3. And, as was mentioned in the informal outline of the algorithm in Section 5.1, for computing only the generators of $[X, P_i]$ it suffices to know the generators of $[X, P_{i-1}]$ and we need not know $[SX, P_{i-1}]$. So a careful (but a little more complicated) implementation of the algorithm can avoid dealing with double and higher suspensions of X .

Acknowledgments

We would like to thank Martin Tancer for useful conversations at early stages of this research, and Peter Landweber for numerous useful comments concerning an earlier version of this paper.

References

- [1] D. J. Anick. The computation of rational homotopy groups is $\#\mathcal{P}$ -hard. *Computers in geometry and topology*, Proc. Conf., Chicago/Ill. 1986, Lect. Notes Pure Appl. Math. 114, 1–56, 1989.
- [2] E. H. Brown (jun.). Finite computability of Postnikov complexes. *Ann. Math. (2)*, 65:1–20, 1957.
- [3] E. B. Curtis. Simplicial homotopy theory. *Advances in Math.*, 6:107–209, 1971.
- [4] H. Edelsbrunner and J. L. Harer. *Computational topology*. American Mathematical Society, Providence, RI, 2010.
- [5] S. Eilenberg. Cohomology and continuous mappings. *Ann. of Math. (2)*, 41:231–251, 1940.
- [6] S. Eilenberg and S. Mac Lane. On the groups $H(\Pi, n)$. II. Methods of computation. *Ann. of Math. (2)*, 60:49–139, 1954.

- [7] P. Franek, S. Ratschan, and P. Zgliczynski. Satisfiability of systems of equations of real analytic functions is quasi-decidable. *Proc. 36th International Symposium on Mathematical Foundations of Computer Science (MFCS)*, to appear, 2011.
- [8] G. Friedman. An elementary illustrated introduction to simplicial sets. Preprint, arXiv:math/0809.4221v3, 2011.
- [9] E. Gawrilow and M. Joswig. polymake: a framework for analyzing convex polytopes. In G. Kalai and G. M. Ziegler, editors, *Polytopes – Combinatorics and Computation*, pages 43–74. Birkhäuser, Basel, 2000.
- [10] P. G. Goerss and J. F. Jardine. *Simplicial homotopy theory*. Birkhäuser, Basel, 1999.
- [11] R. González-Díaz and P. Real. Computation of cohomology operations of finite simplicial complexes. *Homology Homotopy Appl.*, 5(2):83–93, 2003.
- [12] R. Gonzalez-Diaz and P. Real. Simplification techniques for maps in simplicial topology. *J. Symb. Comput.*, 40:1208–1224, October 2005.
- [13] A. Hatcher. *Algebraic Topology*. Cambridge University Press, Cambridge, 2001. Electronic version available at <http://math.cornell.edu/hatcher#AT1>.
- [14] S. Hu. *Homotopy theory*. Academic Press, New York, 1959.
- [15] D. N. Kozlov. Chromatic numbers, morphism complexes, and Stiefel–Whitney characteristic classes. In *Geometric Combinatorics (E. Miller, V. Reiner, and B. Sturmfels, editors)*, pages 249–315. Amer. Math. Soc., Providence, RI, 2007. Also arXiv:math/0505563.
- [16] J. Matoušek. *Using the Borsuk-Ulam theorem (revised 2nd printing)*. Universitext. Springer-Verlag, Berlin, 2007.
- [17] J. Matoušek, M. Tancer, and U. Wagner. Hardness of embedding simplicial complexes in \mathbb{R}^d . *J. Eur. Math. Soc.*, 13(2):259–295, 2011.
- [18] J. P. May. *Simplicial Objects in Algebraic Topology*. Chicago University Press, 1967.
- [19] R. E. Mosher and M. C. Tangora. *Cohomology operations and applications in homotopy theory*. Harper & Row Publishers, New York, 1968.
- [20] A. Nabutovsky and S. Weinberger. Algorithmic unsolvability of the triviality problem for multidimensional knots. *Comment. Math. Helv.*, 71(3):426–434, 1996.
- [21] A. Nabutovsky and S. Weinberger. Algorithmic aspects of homeomorphism problems. In *Tel Aviv Topology Conference: Rothenberg Festschrift (1998)*, volume 231 of *Contemp. Math.*, pages 245–250. Amer. Math. Soc., Providence, RI, 1999.
- [22] D. C. Ravenel. *Complex Cobordism and Stable Homotopy Groups of Spheres (2nd ed.)*. Amer. Math. Soc., 2004.
- [23] A. Romero, J. Rubio, and F. Sergeraert. Computing spectral sequences. *J. Symb. Comput.*, 41(10):1059–1079, 2006.

- [24] J. Rubio and F. Sergeraert. Constructive algebraic topology. *Bull. Sci. Math.*, 126(5):389–412, 2002.
- [25] J. Rubio and F. Sergeraert. Algebraic models for homotopy types. *Homology, Homotopy and Applications*, 17:139–160, 2005.
- [26] J. Rubio and F. Sergeraert. Postnikov “invariants” in 2004. *Georgian Math. J.*, 12:139–155, 2005.
- [27] J. Rubio and F. Sergeraert. Constructive homological algebra and applications. Lecture notes for the 2006 Genova Summer School. Available at <http://www-fourier.ujf-grenoble.fr/~sergerar/Papers/>, 2006.
- [28] R. Schön. Effective algebraic topology. *Mem. Am. Math. Soc.*, 451:63 p., 1991.
- [29] A. Schrijver. *Theory of linear and integer programming*. John Wiley & Sons, Inc., New York, NY, USA, 1986.
- [30] F. Sergeraert. Effective exact couples. Preprint, Univ. Grenoble, available at <http://www-fourier.ujf-grenoble.fr/~sergerar/Papers/>.
- [31] F. Sergeraert. The computability problem in algebraic topology. *Adv. Math.*, 104(1):1–29, 1994.
- [32] A. B. Skopenkov. Embedding and knotting of manifolds in Euclidean spaces. In *Surveys in contemporary mathematics*, volume 347 of *London Math. Soc. Lecture Note Ser.*, pages 248–342. Cambridge Univ. Press, Cambridge, 2008.
- [33] J. R. Smith. m -structures determine integral homotopy type. Preprint, arXiv:math/9809151v1, 1998.
- [34] R. I. Soare. Computability theory and differential geometry. *Bull. Symbolic Logic*, 10(4):457–486, 2004.
- [35] E. H. Spanier. *Algebraic topology*. McGraw Hill, 1966.
- [36] N. E. Steenrod. Products of cocycles and extensions of mappings. *Annals of Mathematics*, 48(2):pp. 290–320, 1947.
- [37] N. E. Steenrod. Cohomology operations, and obstructions to extending continuous functions. *Advances in Math.*, 8:371–416, 1972.
- [38] A. Storjohann. Near optimal algorithms for computing Smith normal forms of integer matrices. In *International Symposium on Symbolic and Algebraic Computation*, pages 267–274, 1996.
- [39] G. W. Whitehead. *Elements of homotopy theory*, volume 61 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1978.
- [40] A. J. Zomorodian. *Topology for computing*, volume 16 of *Cambridge Monographs on Applied and Computational Mathematics*. Cambridge University Press, Cambridge, 2005.